3509 Dynamical Systems Notes

Based on the 2010 autumn lectures by Dr K M Page

0. SOME MOTIVATION

Examples: The Mandelbrot Set

Its boundary displays complicated structure at all length scales — i.e. it's a fractal.
It's generated by a simple iteration, the set $z \in \mathbb{C}$ for which the sequence

$$z_0 = z_0$$
$$z_{n+1} = z_n^2 + z_0 \qquad n = 0, 1, 2, \ldots$$

remains bounded as $n \to \infty$.
The sequence can converge, or not.

e.g. $z = 0 \Rightarrow z_n = 0$

e.g. $z = i \Rightarrow i, \underbrace{i-1, -i}, \underbrace{i-1, -i}, i-1, \ldots$

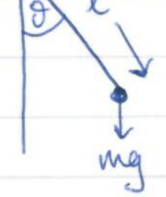
oscillating between 2 values

[You can check that the Mandelbrot set restricted to the real line is just $[-2, 1/4]$.]

In the picture, the black glob is part of the Mandelbrot set. If $|z_n| > 2$, the sequence will diverge.

http://www.math.utah.edu/~pa/math/mandelbrot/mandelbrot.html

Physics



A simple swinging pendulum has eq^n of motion $ml\ddot{\theta} = -mg\sin\theta$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

non-linear because of sine term.

Convection

In 1963, Lorenz found a simplified model of convection rolls in the atmosphere.

- Described by eqⁿs which exhibited chaotic motion on a strange attractor.
- Solution never settled to an equilibrium or a periodic state
- Continued to oscillate in an irregular way
- Behavior was very different starting from nearby initial conditions.
- $\Rightarrow$ the system is unpredictable.

$$\frac{dX}{dt} = S(Y-X)$$

$$\frac{dY}{dt} = RX - Y - XZ$$

$$\frac{dZ}{dt} = XY - bZ$$

Biology

Equations to describe the size of a population of animals each year (e.g. the logistic map - week 5) can also show chaotic dynamics.

Chemistry

Equations describing concentrations of reacting chemicals and how they change in time also make nonlinear dynamical systems.

Economics and Social Sciences

The dynamics of populations of individuals employing certain strategies in games are described by dynamical systems – e.g. frequencies of cooperators and defectors in an evolutionary Prisoners Dilemma.

1. KEY CONCEPTS

1. What is a dynamical system?

:STATE ⟶ PHASE

A dynamical system consists of
(1) a space (the state space or phase space)
(2) a rule describing the evolution of any point in that space.

The state of the system is the set of quantities which we consider interesting or important about the system and the state space is the set of all possible values of those quantities.

There are 2 fundamental 'pictures' associated with dynamical systems depending on whether <u>time</u> is <u>discrete</u> or <u>continuous</u>.

In the first case, <u>the evolution will be defined by a</u> <u>map</u>.
In the second case, ——— " ——— <u>set</u> of differential eq^{n}s.

In this course, we will be interested in ODEs, but not PDEs.
We will always assume the evolution is deterministic.

When time is discrete, we get a map of the form

$$x_{k+1} = f(x_k)$$

(x_k means the value of the variable x at the k^{th} time step.)

The equation tells us that the state of the system at time $k+1$ is given by the f^n f applied to the state of the system at time k. Both x and $f(x)$ live in the <u>state space</u>.

Notation: Given two maps f and g, where the domain of f contains the range of g, $f \circ g$ means $f(g(\cdot))$ — i.e. the composition of the 2 maps.

If f defines a dynamical system, so its domain contains its range, then

$$f^2(x) = f(f(x))$$
$$f^k(x) = \underbrace{f \circ f \circ \cdots \circ f}_{k.}(x)$$

If f is invertible,

$$f^{-k} = \underbrace{f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}}_{k}$$

When a map is not invertible,

$f^{-k}(x)$ is the set $\{y : f^k(y) = x\}$

This set can be empty.

When time is continuous we get a diff-eq^n of the form

$$\dot{x} = f(x,t)$$

a vector field.

(which tells us how the state of the system is changing at a given time).

2. Basic ideas

Several notions are common to both discrete-time and cts-time dyn. sys's. Assume that the state space is $\mathbb{R}^n$.

(1) Solutions A solution is a $f^{\underline{n}}$ of time $g(t)$ (continuous case) or $g(k)$ (discrete case) which satisfies the differential $eq^{\underline{n}}$ or map.

If the $f^{\underline{n}}$ $f(x)$ is invertible, a solution of $x_{k+1} = f(x_k)$ is any $f^{\underline{n}}$ $\phi : \mathbb{Z} \to \mathbb{R}^n$ satisfying

$$\phi(k+1) = f(\phi(k)).$$

If f is noninvertible, then the solution is only defined for $k \geq 0$, i.e. $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$.

A solution to the ODE $\dot{x} = f(x,t)$ on $\mathbb{R}^n$ is a real $f^{\underline{n}}$ ϕ satisfying

$$\frac{d}{dt}\phi(t) = f(\phi(t), t) \qquad \phi : \mathbb{R} \to \mathbb{R}^n.$$

We will see later that the solution need not be defined $\forall\, k \in \mathbb{R}$.

(2) Orbits An orbit is the image of a solution. In other words, it is the path in phase space traced out by a solution.

$Def^{\underline{n}}$: forward orbit (map): for a map f, the forward orbit of the point x is the set $\{x, f(x), f^2(x), \dots\}$

$Def^{\underline{n}}$: backward orbit (map): for invertible map f, the bkwd orbit of the pt x is the set $\{x, f^{-1}(x), f^{-2}(x), \dots\}$

Defⁿ: orbit (map): In general the orbit of a point means the union of fwd & bwd orbits, if the latter exists.

Now consider an ODE $\dot{x} = f(x)$, which has solution $\phi(t)$ with $\phi(0) = x_0$.

Defⁿ: forward orbit (ODE): The forward orbit of x_0 is the set $\{\phi(t) : t \geq 0\}$

Defⁿ: backward orbit (ODE): The backward orbit of x_0 is the set $\{\phi(t) : t \leq 0\}$

(3) Limit sets

These are special sets of orbits. They are, roughly speaking, sets of orbits which attract/repel orbits, ie. objects towards which objects tend in fwd/bkwd time.

(4) Stability

Stability tells us about how nearby solⁿs (orbits) behave. Roughly speaking, if nearby solⁿs (or orbits) stay close to a given solⁿ (or orbit) then that solⁿ/orbit is stable.

(5) Basin of attraction

The subset of phase space which is attracted to a particular limit set is the basin of attraction of the limit set.

(6) Invariant sets

Invariant sets are regions in phase space which contain complete orbits — i.e. if they contain a single point in the orbit, then they contain the whole orbit. Limit sets are invariant.

Illustrative example

Collatz map (discrete dyn. system).

$$f(x) = \begin{cases} 3x+1 & x \text{ odd} \\ x/2 & x \text{ even} \end{cases}$$

State space $\mathbb{N}$.

Orbit: Try e.g. $x_0 = 1$: $1 \to 4 \to 2 \to 1$

$x_0 = 3$: $3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

Limit set : $\{1, 2, 4\}$

Conjecture: $\forall x_0$, the sequence x_n reaches 1 and hence converges to the limit set $\{1,2,4\}$ (unproven, actually !!)

Try $x_0 = 27$ for jokes.

2.1 Fixed/periodic points of maps and equilibria of ODEs

A very special kind of limit set is a fixed point — a point which doesn't move under the dynamics.

Defn: fixed point (map): A fixed point of a map $x_{n+1} = f(x_n)$ is a point x which satisfies $f(x) = x$.

Defn: equilibrium (ODE): An equilibrium in an ODE system $\dot{x} = f(x,t)$ is a point satisfying $f(x,t) = 0 \quad \forall t$.

We will later see that an equilm for an ODE is actually a fixed pt of the associated flow (defined in Week 6).

Defn: periodic point of a map: A periodic point of period n for a map f is simply a fixed point of f^n, i.e. it satisfies $f^n(x) = x$.

If n is the smallest natural no. for which $f^n(x) = x$, n is the prime period of x

All statements about period n points can be regarded as statements about fixed points of f^n.

<u>Def^{n}:</u> <u>eventually periodic point of a map:</u>

A point is eventually periodic if it is not itself periodic, but is eventually mapped by f onto a periodic point of f. Only noninvertible maps have eventually periodic points.

<u>Examples</u> :

1 a) $x_{n+1} = \frac{1}{2}x_n$ fixed pt: 0

b) $x_{n+1} = 2x_n$ fixed pt: 0

a) $x_n = \frac{1}{2^n}x_0 \to 0$ as $n \to \infty$ STABLE

b) $x_n = 2^n x_0 \to \infty$ as $n \to \infty$ UNSTABLE

2. $\theta_{n+1} = \theta_n + \alpha \mod 2\pi$ ("Twist map")

State space is the unit circle

a) $\alpha = \pi/3$

- no fixed points
- all pts are periodic, prime per. 6

b) $\alpha = \pi r$, r irrational

- no periodic pts
- no eventually periodic pts

If θ is eventually periodic

$f^m(\theta)$ is periodic with period n for some $m \in \mathbb{N}$

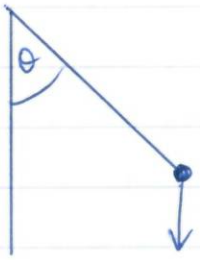
$f^{n+m}(\theta) = f^m(\theta)$, some $m, n \in \mathbb{N}$

$\theta + (n+m)r\theta = \theta + mr\pi \mod 2\pi$

$\Rightarrow nr\pi = 0 \mod 2\pi$

$nr = 0 \mod 1$ ※

State spaces



The state of a pendulum is fully described by

- its angle θ to the vertical, and
- its ang. velocity $\dot{\theta}$.

What does the state space look like?

θ lies on circle

$\dot{\theta}$ lies in real n°s

State space is an infinite cylinder: $(-\pi, \pi] \times \mathbb{R}$

3. Qualitative approach to functions

Often we will need to understand something about the behaviour of a f? without knowing everything about it.

Here we have some questions we might want to ask:

- is it continuous?
- is it differentiable?
- is it invertible?
- does it have a zero?
- does it have a minimum/maximum?
- is it periodic?
- does it satisfy some particular criterion? (e.g. order preserving i.e. $x \leq y \Rightarrow f(x) \leq f(y)$)

<u>Def^n</u>: <u>continuous</u>: a f^n $f(x)$ is continuous at x_0 if
(1) $f(x_0)$ is defined so $x_0 \in D(f)$
(2) $\lim_{x \to x_0} f(x)$ exists for $x \in D(f)$ and any sequence approaching x_0
(3) $\lim_{x \to x_0} f(x) = f(x_0)$ for all these sequences

Suppose we are considering real f^ns unless otherwise stated.

What is the domain of def^n of
(a) $f(x) = x^3$ ▷ $\mathbb{R}$
(b) $f(x) = \frac{ax}{b+x}$ ▷ $\mathbb{R} \setminus \{-b\}$

A f^n is continuous on an open interval I iff it is cts at each point of I

Is (a) continuous on its domain of definition? ▷ Yes.
Is (b) continuous on its domain of definition? ▷ Yes!

<u>Def^n</u>: <u>invertible</u>: a f^n is invertible (bijective) if it is
- one-to-one (injective), and
- onto (surjective).

Functions which are not invertible can have set theoretic 'inverses' - i.e. $f^{-1}(y) = \{x : f(x) = y\}$.

Be careful! If you have f^{-1}, doesn't necessarily mean it's invertible!

Are these invertible on $\mathbb{R}$?
(a) $f(x) = x^3$ ▷ Yes
(b) $f(x) = x^2$ ▷ No

> Def^n homeomorphism: The f^n $f(x)$ is a homeomorphism if $f(x)$ is one-to-one, onto, and continuous, and $f^{-1}(x)$ is continuous.

Homeomorphism will be important when we discuss conjugacy of maps

$f:[0,2\pi) \to S$ $\quad f(\phi) = (\cos\phi, \sin\phi)$.
Is it a homeomorphism?

Def^n C^r: We say a f^n is C^r if it is r times diff^{ble} and there first r derivatives are continuous

~~Finding zeroes of a f^n of many variables~~

We will often have to find zeroes of a f^n. Consider a f^n $g: \mathbb{R}^n \to \mathbb{R}^n$, finding zeroes may be difficult. Graphical methods may be very useful.

Example: Consider the function

$$f(x) = \begin{cases} x & x \geq 1 \\ x^2 & x < 1 \end{cases} \qquad f:\mathbb{R}\to\mathbb{R}$$

Is it differentiable at $x=1$?
No
(but it is cts).

4. Some basic useful theorems

We now state some basic theorems of analysis, which will be used later in the course.

Note that th

Note that although the IVT and MVT are 1-D results, they can still be very useful and can sometimes be used to derive results in higher dimension.

4.1 Intermediate Value Theorem (IVT)

Thm: Suppose $f : [a,b] \to \mathbb{R}$ is cts.
Suppose that $f(a) = u$
$f(b) = v$.
Then for any z between u and v,
$\exists\ c \in (a,b)$ s.t. $f(c) = z$.

Proof: See Yiannis. $\square$

Notation: $f(I) = \{ y : f(x) = y,\ x \in I \}$

Example: The IVT implies that if $I \subset \mathbb{R}$ is a closed, bounded interval, and cts $f : I \to \mathbb{R}$ satisfies $f(I) \supseteq I$, then f has a fixed pt in I.

Proof: Let $I = [a,b]$. Consider $\alpha \in f^{-1}(a)$, and
$\beta \in f^{-1}(b)$.
(these 2 sets are not empty $\because f(I) \supseteq I$).

Consider the f^n $g(x) = f(x) - x$.

We have $g(\alpha) = \overbrace{f(\alpha)}^{a} - \alpha \leq 0$
$g(\beta) = f(\beta) - \beta \geq 0$

So by IVT, $\exists\ p \in [\alpha, \beta]$ or $[\beta, \alpha]$ s.t. $g(p) = 0$.
$\Rightarrow f(p) = p$ $\square$.

4.2 Mean Value Theorem (MVT)

Thm: Suppose $f : [a,b] \to \mathbb{R}$ is C^1.
Then $\exists$ $c \in [a,b]$ s.t.

$$f(b) - f(a) = f'(c)(b-a).$$

Proof: More first year stuff $\square$.

4.3 Convergence theorems

A lot of basic ideas in dynamical systems rely on the Bolzano-Weierstraß theorem,

↳ each bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

A sequence of real n°s x_k is monotone if

$x_k \leq x_{k+1}$ $\forall k$ (nondecreasing)

$x_k \geq x_{k+1}$ $\forall k$ (nonincreasing)

Monotone convergence theorem

↳ a monotone sequence in a compact subset of the real line converges.

Example: Show that the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x}$ has a fixed point in $[0,1]$.

Proof: Let $g(x) = f(x) - x$.

$g(0) = 1$, $\quad g(1) = \frac{1}{e} - 1 < 0$

g is continuous.
Then apply the IVT $\Rightarrow$ $\exists p \in [0,1]$ s.t. $g(p) = 0$. $\square$

Example: Monotone increasing map.
What can happen?

Answer: So $f(x) > f(y)$ if $x > y$.

If $f(x_0) > x_0$, then $f^2(x_0) > f(x_0)$

$\cdots f^n(x_0) > f^{n-1}(x_0) \cdots$

i.e. the sequence $x_0, f(x_0), f^2(x_0), \ldots$
is monotone increasing
and $\therefore \longrightarrow \infty$
or cvgs to upper bound monotonically.

If $f(x_0) < x_0$, then $f^n(x_0) < f^{n-1}(x_0)$

i.e. the sequence $x_0, f(x_0), f^2(x_0), \ldots$
is monotone decreasing
and $\therefore \rightarrow -\infty$
or cvgs to lower bound monotonically

If $f(x_0) = x_0$, then the sequence $x_0, f(x_0), \ldots$
stays at the fixed point x_0.

Further examples: (i) $f: \mathbb{Z}_{10} \to \mathbb{Z}$ $\quad f(x) = 2x \pmod{10}$

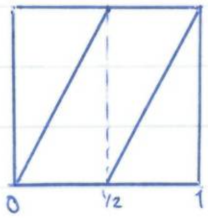
limit set: $\quad f(0) = 0 \quad \rightarrow \{0\}$

$f(1): 2, 4, 8, 6, \rightarrow$

~~f(3)~~ $\searrow$ $\{2, 4, 8, 6\}$ period 4 orbit.

basin of attraction. $\{0, 5\}$ $\quad \mathbb{Z}_{10} \setminus \{0, 5\}$

<u>Doubling map:</u> $T : [0,1) \to [0,1)$
$T(x) = 2x \mod 1$



fixed pts: 0.
$(2x = x \mod 1)$
$(x = 0 \mod 1)$
$(x = 0)$.

<u>Periodic and eventually periodic points</u>

Periodic pt: $f(x) = 2x \mod 1$
$f^n(x) = x$ for a pt of period n.

$2^n x = x \mod 1$
$(2^n - 1)x = 0 \mod 1$ all are 0 mod 1
$x = \dfrac{m}{2^n - 1}$ $m = 0, 1, \ldots, 2^n - 2$ (these are the pts of period n)

$\because$ the next one, 2^n-1, gives $x=1$ which is too big

$n=1$ is a fixed pt and this means $x = 0 \mod 1$.

$n>1$, clearly x must be a rational with an odd denominator (in its reduced form)

Claim: Any rational x with an odd denominator satisfies $(2^n - 1)x = 0 \mod 1$
(and hence is a periodic pt with period n)
for some $n \in \mathbb{N}$.

Proof: Let $x = \frac{r}{k}$, $r, k \in \mathbb{N}$, k odd

Euler's theorem $\xRightarrow{k \text{ odd}}$ $2^{\Phi(k)} = 1 \mod k$,
where $\Phi(k)$ ← Totient fn is the nº of positive integers $\leq k$ that are coprime to k.

$2^{\Phi(k)} - 1 = km$ for some $m \in \mathbb{N}$

$\Rightarrow (2^{\Phi(k)} - 1)\frac{1}{k} = 0 \mod 1$

$\Rightarrow (2^{\Phi(k)} - 1)\frac{r}{k} = 0 \mod 1$

$\Rightarrow \frac{r}{k}$ is periodic under the doubling map with period $\Phi(k)$.

$\Phi(7) = 6$.
(e.g. $k = 7$, $\frac{1}{7}$ has prime period 3).

$\Rightarrow$ the periodic pts of the doubling map with period > 1 are precisely the rational nºs with odd denominators (in reduced form).

<u>Eventually periodic pts.</u>

$\exists\, p, n \in \mathbb{N}$ (>0) s.t. $2^p x = 2^{n+p} x \mod 1$.

$\Leftrightarrow x(2^{n+p} - 2^p) = 0 \mod 1$

$\Leftrightarrow x = \frac{m}{2^p(2^n - 1)}$ $\quad m = 0, 1, \ldots, 2^{n+p} - 2^p - 1$

$\Rightarrow$ x must clearly be rational.
As we have established that any ~~ra~~ rational nº with odd denominator can be expressed as $\frac{m}{2^n - 1}$, clearly any rational with even denominator (in reduced form) can be expressed as $\frac{m}{2^p(2^n-1)}$ with $p \geq 1$.

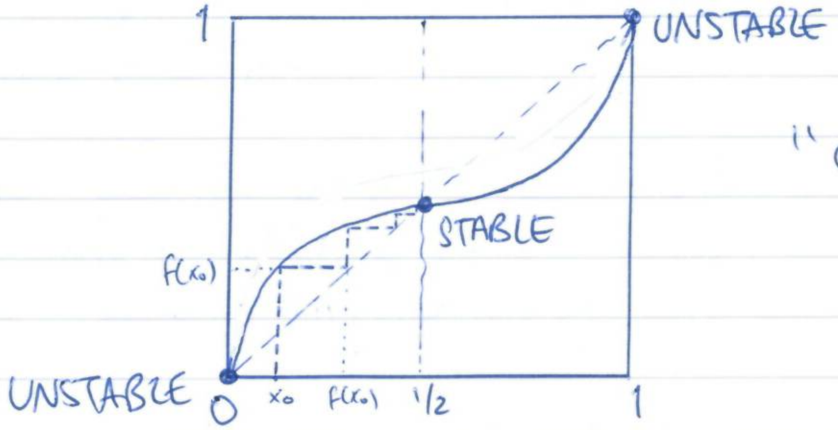
$\Rightarrow$ the eventually periodic points are the rational nºs with even denominator (in reduced form).

Introduction to discrete-time systems

Graphical Analysis

Graphical analysis is a useful technique which allows you to explore discrete-time dynamical systems in 1D. Sometimes it allows you to characterise the behaviour of the system completely.

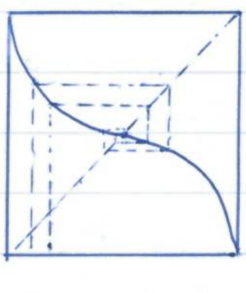
Consider the system $x_{n+1} = f(x_n)$
where $f : [0,1] \to [0,1]$ has graph shown below:



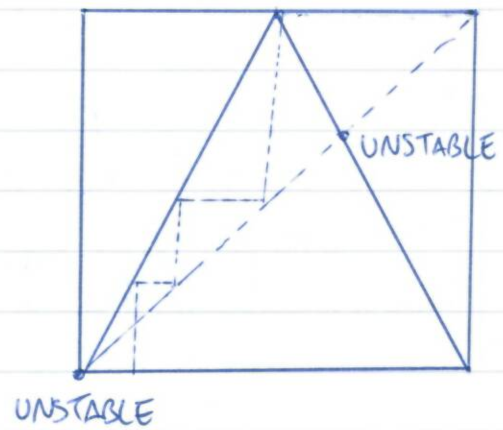
"cobweb diagram"

From the graph alone it is possible to conclude that there are three fixed points, and that all points except the fixed pts at $x = 0, 1$ are attracted to fixed pt at $x = \frac{1}{2}$.

Graphical analysis is an important tool for trying to understand 1D maps before trying to prove anything.



← another map



"tent map"

$T : [0,1] \to [0,1]$

$$T(x) = \begin{cases} 2x & x \le 1/2 \\ 2(1-x) & x > 1/2 \end{cases}$$

Different possible cases: (i) move directly twd fixed pt
(ii) move directly away from fixed pt
(iii) move toward fixed pt but oscillating either side of it
(iv) move away from fixed pt but oscillating either side of it.

We will show how to determine which of these options applies to a given fixed point of a given map f.

Stability, instability and hyperbolicity of fixed points of 1D maps

The basic idea, in any dimension, is that a fixed pt p is "stable" if the orbits of all the points near p never move far away from p.

Def^n: stable: Let p be a periodic pt of period n for a map f. The point p is stable if, given any neighbourhood V of p, we can find a neighbourhood U of p s.t.

$$\forall x \in U \quad \forall m \geq 0, \quad f^{nm}(x) \subset V.$$

Another kind of stability is:

Def^n: asymptotically stable. Let p be a periodic point of period n of the map f.

The point p is asymptotically stable if $\exists$ a neighborhood U of p s.t. $\lim_{m\to\infty} f^{nm}(x) = p \ \forall x \in U$.

In other words, p is asymptotically stable if the orbit of every point in some neighborhood of p eventually converges to p.

When a map is C^1, we can often get information about the stability of an object from the derivative of a map evaluated at that object.

Such analysis is called linear stability analysis because the derivative is a linear map.

The next two results show us why conditions on the derivative at the fixed pt tell us about local stability of the fixed point.

Thm: linear asymptotic stability.

Let p be a fixed point of a 1D C^1 map, satisfying $|f'(p)| < 1$. Then p is asymptotically stable. In other words, there is an open interval U about p s.t. if $x \in U$ then $\lim_{n\to\infty} f^n(x) = p$.

Sketch proof: Since $f \in C^1$, $\exists\ \varepsilon > 0$ s.t. $|f'(x)| < A < 1$ for $x \in [p-\varepsilon, p+\varepsilon]$.

By the MVT, $|f(x)-f(p)| \leq A|x-p| < |x-p| \leq \varepsilon$
Hence $f(x)$ is closer to p than x, and still lies in $[p-\varepsilon, p+\varepsilon]$. Via an inductive argument,

$$|f^n(x)-p| \leq A^n|x-p|$$

So that $f^n(x) \to p$ as $n \to \infty$ (since $A^n \to 0$) $\square$.

This theorem generalises immediately to periodic points: if we are interested in the stability of an orbit of period k, we simply apply the above argument to f^k rather then f.

<u>Def^n:</u> <u>unstable:</u> not stable (serious).

<u>Thm: linear instability</u>

Let p be a fixed pt of a 1D C^1 map so that $|f'(p)| > 1$. Then there is an open interval U about p s.t. $\forall x \in U \setminus \{p\}$, $\exists$ $k \geq 1$ s.t. $f^k(x) \notin U$.

<u>Proof</u> left as exercise. $\square$

The theorem tells us that the condition $|f'(p)| > 1$ means that all points diverge from p.
This generalises to periodic points.

<u>Def^n:</u> <u>hyperbolic</u>: Let p be a periodic pt of prime period n for a map f. The point p is hyperbolic if $|(f^n)'(p)| = 1$.

Note that if a fixed pt is hyperbolic, it can be asymptotically stable <u>or</u> unstable.

Note we only have defined hyperbolicity for differentiable maps.

Using the chain rule

If we are interested in the stability of a period k point p, we need to evaluate $(f^k)'(p)$.
Usually it is easier to use

$$(f^k)'(p) = f'(f^{k-1}(p))\, f'(f^{k-2}(p)) \cdots f'(f(p))\, f'(p)$$

i.e. the derivative of f^k at p is the product of the derivative of f at points $p, f(p), f^2(p), \ldots, f^{k-1}(p)$.

Usually it's much easier to evaluate f' at k points along the orbit of p rather than $(f^k)'(p)$.

Example of using chain rule for stability of a 2-cycle

$f(x) = 1 - x^2$.

Fixed pts: $1 - x^2 = x$

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

roots of these are roots of these

Period 2 cycle: $f^2(x) = 1 - (1-x^2)^2$

$$= 1 - (1 - 2x^2 + x^4)$$

$$= -x^4 + 2x^2$$

$$f^2(x) = x \Rightarrow x^4 - 2x^2 + x = 0$$

i.e. $x(x^3 - 2x + 1) = 0$.

$$\Rightarrow x = 0, 1, \frac{-1 \pm \sqrt{5}}{2}$$

Is it stable? $\quad f'(x) = -2x.$

and $(f^2)'(x) = -4x^3 + 4x$

or $(f^2)'(p) = f'(p)\, f'[\underbrace{f(p)}_{=0}] < 1$

$\Rightarrow$ stable.

Period 3 implies all other periods.

This is our first main result of the course

Notation: if A & B are two intervals, then $A \to B$ or $A \xrightarrow{f} B$ will mean $f(A) \supseteq B$ ("A covers B").

Similarly $A \xrightarrow{f^n} B$ means $f^n(A) \supseteq B$.

We will write $A \multimap B$ or $A \overset{f}{\multimap} B$ to mean $f(A) = B$ ("A maps onto B")

f(A) = B

Similarly $A \overset{f^n}{\multimap} B$ means $f^n(A) = B$.

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. Suppose f has a periodic point of prime period 3. Then f has periodic points with prime period n for every $n > 0$.

Preliminary 1: If I is a closed interval and $I \xrightarrow{f^n} I$ then f^n has a fixed point in I. This is a corollary of IVT and has been proven already.

Preliminary 2: Suppose $A \to B$ for two closed intervals A and B. Then there is at least one closed subinterval $A_0 \subseteq A$ s.t. $A_0 \multimap B$, ie. $\exists$ a sub-interval of A which maps exactly onto B.

Proof: Suppose $B = [c, d]$

$$A \xrightarrow{f} B \Rightarrow \begin{array}{l} A \cap f^{-1}(c) \neq \emptyset \\ A \cap f^{-1}(d) \neq \emptyset \end{array}$$

$f^{-1}(c) \subseteq A$, which compact and f is continuous, so $f^{-1}(c)$ has a lowest element. Likewise $f^{-1}(d)$ has a lowest el.

Consider $f^{-1}\{c, d\} = f^{-1}(c) \cup f^{-1}(d)$.

Suppose the lowest element of $f^{-1}\{c, d\}$ maps to c (the other case follows similarly), then let Y be the lowest element of $f^{-1}(d)$.
Let X be the highest element in $f^{-1}(c)$ which is less than Y. (again, it must exist, since f is cts and $A_{\leq Y}$ is compact).

Now $f(X) = c$ and $f(Y) = d$
and $\nexists\, Z \in [X, Y]$ s.t. $f(Z) = c$ or d

$\Rightarrow$ by IVT, $B \subseteq f[X, Y]$.
and $\nexists\, u \in [x, y]$ s.t. $f(u) > d$ (or there would be $v \in [x, u]$ s.t. $f(v) = d$).
Likewise $\nexists\, w \in [x, y]$ s.t. $f(w) < c$.
$\Rightarrow [x, y] \multimap B$
and $[x, y] \subseteq A$ $\square$

Preliminary 2 implies the following:

Suppose $A_0, A_1, \dots, A_n$ are closed intervals and $A_i \to A_{i+1}$ for $i = 0, \dots, n-1$.
Then $\exists$ at least 1 subinterval J_1 of A_0 satisfying $J_1 \multimap A_1$.

There is a similar subinterval of A_1 which is mapped onto A_2 and therefore there is a subinterval $J_2 \subseteq J_1$, s.t. $f(J_2) \subseteq A_1$, and $J_2 \overset{f^2}{\multimap} A_2$.

Continuing, we find an interval $J_n \in A_0$ s.t. $f^n(J_n) \subseteq A_i$ for $i = 1, \dots, n-1$ and $f^n(J) = A_n$.

<u>Sketch proof of the theorem</u>

Let $a, b, c \in \mathbb{R}$ be the three points of period 3, with $a < b < c$. Suppose for definiteness that $f(a) = b$, $f(b) = c$, $f(c) = a$.

(The other possibility with $f(a) = c$ can be dealt with similarly)

Let $A = [a, b]$ and $B = [b, c]$.

(1) Fixed point: by our assumptions, $A \to B$ and $B \to \overbrace{A \cup B}^{=[a,c]}$,
The second of these implies that $B \to B$, and so by preliminary 1, there must be a fixed point of f in B.

(2) $A \to B \Rightarrow \exists\ A_0 \overset{f}{\multimap} B \xrightarrow{f} A \cup B$

$A_0 \multimap B \xrightarrow{f} A_0$

$\Rightarrow A_0 \xrightarrow{f^2} A_0 \Rightarrow$ there is a fixed pt of f^2 in A_0

$\Rightarrow \exists$ period-2 pt in A_0.

and $f(x) \in B$ (and can't be b, so is not in A and $\therefore$ not in A_0).

So x has prime period 2.

(3) $n \geq 4$

We use preliminary 2 repeatedly.

Let $B_1 \subseteq B$ be an interval s.t. $B_1 \multimap B$.

Let $B_2 \subseteq B_1$ . . . . $B_2 \multimap B_1$ etc.

$\Rightarrow B_{n-2} \overset{f}{\multimap} B_{n-3} \overset{f}{\multimap} \cdots \multimap B_2 \overset{f}{\multimap} B_1 \overset{f}{\multimap} B$

or $B_{n-2} \overset{f^{n-2}}{\multimap} B$.

Since $B \to A$, we have $B_{n-2} \overset{f^{n-2}}{\multimap} B \to A$

i.e. $B_{n-2} \xrightarrow{f^{n-1}} A$.

Thus by preliminary 2, $\exists$ a subinterval $B_{n-1} \subseteq B_{n-2}$ s.t. $B_{n-1} \overset{f^{n-1}}{\multimap} A$.

But since $A \to B$, we have

$B_{n-1} \overset{f^{n-1}}{\multimap} A \xrightarrow{f} B$ i.e. $B_{n-1} \xrightarrow{f^n} B$

which clearly implies $B_{n-1} \xrightarrow{f^n} B_{n-1}$

Thus f^n has a fixed point in B_{n-1}.

But the first $n-2$ iterates of p lie in B and the $(n-1)^{th}$ lies in A. Assuming pts do not lie in $A \cap B$ then p must have prime period n.

The curious case of an iterate on $A \cap B = \{b\}$

Suppose $\exists r \in \{0, 1, \dots, n-1\}$ s.t. $f^r(p) = b$.

Now $f^n(p) = p \Rightarrow f^{n-r}(b) = p \Rightarrow p \in \{a, b, c\}$.

$$\Rightarrow a \in \{p, f(p), f^2(p)\}$$

but $a \notin B$ and
$$p \in B_{n-1} \subseteq B$$
$$f(p) \in B_{n-2} \subseteq B$$
$$f^2(p) \in B_{n-3} \subseteq B \qquad (n \geqslant 4) \qquad \#$$

$\Rightarrow$ there is no iterate on the boundary $A \cap B$ $\square$.

Sarkovskii's thm: statement, but not proof

That period 3 implies all other periods is a special case of a more general theorem.

Consider the following ordering of the natural n°s:

$$\underbrace{3 \rhd 5 \rhd 7 \rhd \cdots}_{\text{odds}} \rhd \underbrace{2\cdot 3 \rhd 2\cdot 5 \rhd \cdots}_{2\times\text{ odds}} \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd \cdots \rhd \underbrace{2^3 \rhd 2^2 \rhd 2 \rhd 1}_{\text{decreasing powers of 2}}$$

"Sarkovskii's ordering of the $\mathbb{N}$s"

Sarkovskii's thm is:

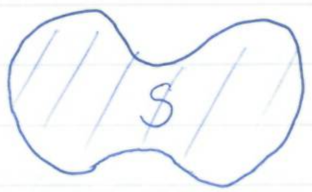
Suppose f is a real, cts f^n and has a periodic pt of prime period k. If $k \rhd \cdots \rhd l$ $(l \in \mathbb{N})$ then f also has a periodic pt of prime period l.

For example, if a cts map on $\mathbb{R}$ has a period 7 orbit, then it must have pts of every period except possibly 3 and 5.

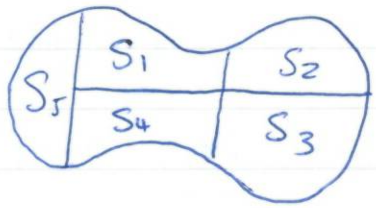
The proof is similar to the period 3 result but quite a bit harder!

Suppose we have a map f acting on an interval I. We can divide the state-space (in this case I) into closed subsets (possibly overlapping at the margins).

In a more general case, suppose f acts on S



We can divide S into closed subsets:



Then for any orbit of f: $x, f(x), f^2(x), \ldots$
we can write down which subset each iterate is in.

This is called the itenerary of x.

e.g. $x \in S_3$, $f(x) \in S_4$, $f^2(x) \in S_1, \ldots$

$\Rightarrow$ itenerary of x is $S_3 S_4 S_1, \ldots$

$\Rightarrow$ there is a map which takes orbits to iteneraries.

Intervals and itineraries: maps with complicated behaviour.

<u>Def^n</u>: <u>Itinerary</u> Suppose we have a map f acting on an interval I. Let A and B be closed subintervals of I. If we say that point $x \in I$ has <u>itinerary</u> $\underline{s}$, where

$$\underline{s} = s_0 s_1 s_2 s_3 \ldots$$

is an infinite sequence of As and Bs, then this means that $x \in s_0$, $f(x) \in s_1$ and in general $f^k(x) \in s_k$.

For example, let $\underline{s} = ABB\ldots$ be the itenerary of x. This means $x \in A$, $f(x) \in B$, $f^2(x) = B$ etc.

<u>Periodic</u> An itinerary $\underline{s}$ is <u>periodic</u> with period n if removing the first n elements of $\underline{s}$ gives us back $\underline{s}$. n is the <u>prime period</u> of the itinerary if n is the smallest integer for which this is true. In general, two different pts may have the same itinerary.

A <u>periodic orbit</u> maps to a <u>periodic itinerary</u> with the same period (so, ruling out intersections the boundary, if an itinerary is aperiodic it cannot correspond to a periodic orbit).

An <u>eventually periodic orbit</u> maps to an <u>eventually periodic itinerary</u>.

Example: itineraries of the doubling map.

$f:[0,1]\to[0,1]$, $f(x) = 2x \bmod 1$.

Let $L = [0, 1/2]$, $R = [1/2, 1]$.

Note $L \to L\cup R$ and $R \to L \cup R$.

Recall all rationals in $[0,1]$ are eventually periodic (indeed, some are periodic).

Write down itineraries of:

(i) $\frac{1}{3}$: $\overset{1/3}{L}\overset{2/3}{R}\overset{4/3=1/3}{L}RLRL\ldots$ periodic (period 2)

(ii) $\frac{1}{5}$: $\overset{\frac15}{L}\overset{\frac25}{L}\overset{\frac45}{R}\overset{\frac35}{R}\overset{\frac15}{L}LRR\ldots$ periodic (period 4)

(iii) $\frac{1}{20}$: $\overset{\frac{1}{20}}{L}\,\overset{\frac{1}{10}}{L}\,\underbrace{\overset{\frac15}{L}\overset{\frac25}{L}\overset{\frac45}{R}\overset{\frac35}{R}}\underbrace{\overset{\frac15}{L}LRR}\ldots$ ev. per. (p. 4)

(iv) $\frac{\sqrt2}{2}$: $\overset{\sqrt2/2}{R}\ \overset{\sqrt2-1}{L}\ \overset{2\sqrt2-2}{R}\,RLRLRLL\ldots$ no apparent pattern (aperiodic)

Note: e.g. an orbit with itinerary $(AAAB)^\infty$ could be a period 4 orbit, but not a fixed pt or a period 2, p.3, p.5 orbit etc. (excluding landing on $A\cap B$).

Recall from the proof of period 3 $\Rightarrow$ all other periods:

Period 3 $\Rightarrow \exists$ A and subintervals of I
s.t. $A \to B$
and $B \to A \cup B$.

(which we represent $\begin{tikzcd} A \arrow[r, bend left] & B \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$)

This allowed us to construct
$$B_{n-1} \xrightarrow{f^{n-1}}\!\!\circ\, A \xrightarrow{f} B \implies \exists \text{ an orbit of prime period } n, \text{ starting in } B_{n-1}.$$

If we have instead $A \to A \cup B$, $B \to A \cup B$ $\quad \begin{tikzcd} A \arrow[loop left] \arrow[r, bend left] & B \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$

then it looks like we could have any itenerary.

Let us take a fixed finite itinerary $s_0 s_1 s_2, \dots, s_n$ with each s_i either A or B.

Since $s_0 \to s_1$ $\exists$ a closed interval $s_0^1 \subseteq s_0$ s.t. $s_0^1 \multimap s_1$.

Similarly $\exists$ a closed interval $s_1^2 \subseteq s_1$ s.t. $s_1^2 \multimap s_2$ and $\exists$ a closed interval $s_0^2 \subseteq s_0$ s.t. $s_0^2 \multimap s_1^2$.

By induction, $\exists$ closed intervals
$s_0^n, s_1^n, \dots, s_{n-1}^n$ s.t. $s_j^n \subseteq s_j$ and $s_0^n \multimap s_1^n \multimap \cdots \multimap s_{n-1}^n \multimap s$

So for a point in s_0^n, its itinerary is $s_0 s_1 \dots s_n$.

So we have proved that if $A \to A \cup B$ and $B \to A \cup B$, then given any finite itinerary $s_0 s_1 \dots s_n$, we can find an orbit with this itinerary.

This can be extended to infinite itineraries:

Thm (two intervals):

Consider a continuous 1D map f on an ~~closed~~ interval I. Suppose that I contains two disjoint closed subintervals A and B s.t. $A \to A \cup B$ and $B \to A \cup B$. Then for any given itinerary we can find a point in $A \cup B$ whose iterates have this itinerary. Further, given any periodic itinerary, we can find a periodic orbit with this itinerary.

Preliminary lemma: Cantor's intersection thm

Let $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$ be a sequence of non-empty compact subsets of $\mathbb{R}$. Then $\bigcap_i K_i$ is not empty.

closed bounded

$$\left[\text{e.g. } \bigcap_{n=0}^{\infty} (n, \infty) = \emptyset \right.$$

not bounded

$$\left. \bigcap_{n=0}^{\infty} (0, \tfrac{1}{n}) = \emptyset \right]$$

not closed

Proof: not given, but basic analysis.

Sketch proof of 2 intervals thm.

Choose an itinerary $\underline{S} = s_0 s_1 s_2 \ldots$ (each s_i is either A or B).

Define $K_0 = s_0$. Since $s_0 \to s_1$, $\exists$ closed interval $K_1 \subseteq K_0$ which maps onto s_1 i.e. $f(K_1) = s_1$.

Similarly since $s_1 \to s_2$, $\exists$ a $K_2 \subseteq K_1$ s.t. $f^2(K_2) = s_2$. So each pt in K_2 lies in s_0, maps into s_1 and then maps into s_2.

Continuing this way, for any sequence $\underline{s}$ we get a sequence of closed intervals $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$ and by Cantor's intersection thm, the sequence has a non-empty intersection, $K_\infty = \bigcap_{i=0}^{\infty} K_i$.

By construction, all points in K_∞ have itinerary $\underline{s}$. So for any itinerary, we can find a point with that itinerary.

Choose a periodic itinerary of period n. This means that $s_n = s_0$. Now carry out the first n steps of the above construction to get a sequence of intervals $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n$.

We get $f^n(K_n) = s_n$. Now $K_n \subseteq K_0 = s_0 = s_n$ so $f^n(K_n) \supseteq K_n$. Thus f^n has a fixed point in K_n.

By choosing the itinerary to have prime period n, the periodic pt we have found must have prime period n. $\square$

The above theorem ensures that a map with intervals A and B satisfying $f(A) \supseteq A \cup B$ and $f(B) \supseteq A \cup B$ has complicated behaviours with countable periodic orbits and uncountable aperiodic orbits.

We don't know in advance about the stability of these orbits or how many pts have the same itinerary. Generally most of the periodic orbits are unstable.

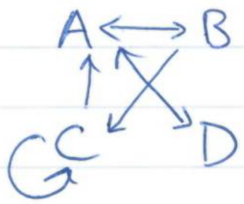
General approach to maps with intervals which cover one another

Consider a map on $\mathbb{R}$ which contains more disjoint intervals, say A, B, C and D satisfying

e.g. $A \to B \cup D$, $B \to A \cup C$, $C \to C \cup A$, $D \to A$.

This generates a directed graph on the four vertices A, B, C, D where two vertices are connected if $X \to Y$, i.e. $f(X) \supseteq Y$.

e.g.



We can define a sequence as 'allowed' if it exists as a path in the graph.

Theorem (many intervals) Consider a 1D map f on an interval I with some set of intervals covering each other and thus generating a directed graph. Corresponding to any path in this graph is an allowed itinerary.

For each allowed itinerary, there is a point with this itinerary. The proof is very similar to the 2 intervals thm.

Maps of the circle

Because of the geometry of the circle, very simple maps of the circle can display complicated behavior.

Exercise Discuss periodic and aperiodic orbits of $f: S^1 \to S^1$ where $f(\theta) = \theta + 2\pi r$ where $0 < r < 1$.
If r is rational, these are called rational rotations, otherwise they're called irrational rotations.

θ is periodic $\qquad \theta + 2\pi n r = \theta \mod 2\pi$
$\qquad\qquad\qquad\qquad nr = 0 \mod 1$

For fixed r, either all θ or no θ are periodic.

If $r \in \mathbb{Q}$ all points are periodic
$r \notin \mathbb{Q}$ all points are aperiodic.
No eventually periodic points.

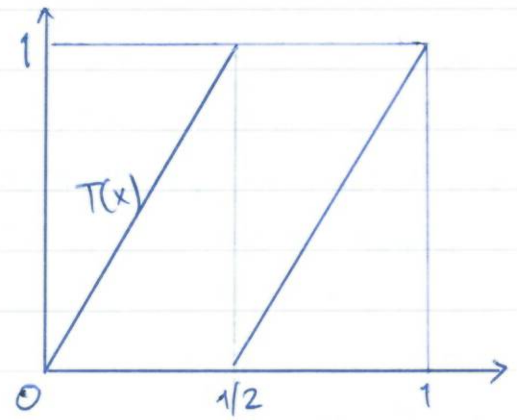
Def^n: dense: A set A is dense in a set B if $B = \bar{A}$, i.e. B is the closure of A.

This is equivalent to: "Given any point in B, we can find a point in A in any open neighbourhood of this point".

Exercise. Show that every point of an irrational rotation has an orbit which is dense in S^1.

Exploration of the doubling map

$T(x) = 2x \bmod 1$ $\quad$ (equivalently $T(\theta) = 2\theta \pmod{2\pi}$ where $\theta \in S^1$)



We can prove that periodic points are dense and so are eventually periodic points. Even eventually fixed points are dense.

To understand the map fully it is convenient to consider $T(x)$ as a map on the half-open interval $[0,1)$ and write it in the form $T(x) = 2x \bmod 1$.

Note that although it looks as though the map is discontinuous, if we regard it as a map of the circle, it is continuous.

Decimals

(a) Note there is ambiguity: $0.10\dot{0} = 0.09\dot{9}$

(b) Explain why n°s with finite decimal representation are dense in $\mathbb{R}$.

↳ For $x \in \mathbb{R}$ and $\varepsilon > 0$, $\exists$ m, sufficiently large, s.t. $10^{-m} < \varepsilon$. Now round x to m decimal places and call this y. y has a finite representation and $|x-y| < 10^{-m} < \varepsilon$ $\Rightarrow$ the set of n°s with finite representation is dense in $\mathbb{R}$.

Note: the set of real n°s with finite decimal representations $\subseteq \mathbb{Q} \Rightarrow \mathbb{Q}$ is dense in $\mathbb{R}$.

Expanding real n°s in different bases

The decimal number $0.a_1a_2a_3\ldots$ $\quad a_i \in \{0, \ldots, 9\}$ has the following meaning:

$$0.a_1a_2a_3\ldots = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots$$

This series converges for any values of a_i because it is dominated by the geometric series $\sum_{n=1}^{\infty} \frac{9}{10^n}$ $(=1)$.

In exactly the same way, we can expand any n° base 2. The n° $0.a_1a_2a_3\ldots$ with $a_i \in \{0,1\}$ means, in base 2

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots$$

e.g. in base 2 the n° 0.11001 is...

$$0.11001 = \frac{1}{2} + \frac{1}{4} + \frac{1}{32} = \frac{25}{32}.$$

In any base there is some ambiguity in how n°s can be written

e.g. $0.6\dot{0} = 0.5\dot{9}$ (base 10)
$0.0\dot{1} = 0.1\dot{0}$ (base 2)

We can avoid this ambiguity in base 2 by insisting that a n° which ends with an infinite string of 1s must be written instead ending in an infinite string of 0s.

Exercise: What is $\frac{1}{7}$ base 5?

$$\frac{1}{7} = \frac{3}{25} + \frac{2}{125} + \frac{4}{625} + \frac{1}{5^5} + \dots$$

$$\frac{1}{7} = 0.\dot{0}3241\dot{2}$$

(can use division
$$7 \overline{) 1.00000000 } \quad 0.03241203$$

Action of the doubling map $T(x)$ on points written in base 2

Let $x = 0.a_1a_2a_3\dots$ in base 2, ie.

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \sum_{n=3}^{\infty} \frac{a_n}{2^n}$$

$$\text{So } 2x = a_1 + \sum_{n=1}^{\infty} \frac{a_{n+1}}{2^n}$$

$$T(x) = 2x \bmod 1$$
$$= \sum_{n=1}^{\infty} \frac{a_{n+1}}{2^n} = 0.a_2a_3a_4\dots$$

So if $x = 0.a_1a_2a_3\dots$
$T(x) = 0.a_2a_3a_4\dots$

Periodic and eventually periodic points

If a point has a binary expansion which repeats after n places, then applying the doubling map n times gives the original point. In other words, such points have period n under the doubling map.

There are 2^n-1 of them and they are dense $\because$ arbitrarily close to a point in $[0,1]$ is a point with periodic binary expansion (or even finite). Eventually periodic points are

$$0.a_1a_2\cdots a_{n-1}\dot{a}_n\cdots\dot{a}_m$$

and are also dense.

NOT KP ON MONDAY

Symbolic dynamics

A shift map is a map on a sequence space that defines the natural generalisation of the doubling map, base 2.

A sequence space $\Sigma_2 = \{ \underline{s} = (s_0 s_1 \ldots s_n) : s_j = 0 \text{ or } 1 \}$

So let's have a shift map $\sigma(s_0 s_1 s_2 \cdots s_n) = s_1 s_2 \cdots s_n$, it removes the first entry.

A limit point of a sequence (x_n) p is a point in a metric space s.t. there is a subsequence $(x_m) \subseteq (x_n)$ s.t. $\lim_{m\to\infty} x_m = p$.

Limit set: A set x has a limit point p.
The limit set of x, $\omega(x)$ is defined as
$\omega(x) = \{ y \in \mathbb{R}^n : \exists$ a sequence (n_j) s.t. $(n_j) \to \infty$ and $f^{n_j}(x) \to y$ as $j \to \infty. \}$

Theorem: Consider a map f acting on $\mathbb{R}^n$ and its subsets. If the orbit of a point p enters and never leaves a closed bounded region in $\mathbb{R}^n$, then it must have a limit point.

Proof: (B-W says every bounded sequence has a cvgt subsequence.)

Every member of $\mathbb{R}^n$ is an infinite set of points
(ie. some number = $A.a_0a_1a_2a_3\dots$)
which is bounded. Thus it has a convergent subsequence.
The point to which it converges is the limit point.

Find the limit set of..

(i) $1, 10, 100, 1000 \dots$ Limit set: $\emptyset$
(ii) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$ limit set: $\{0\}$.

If f is an invertible map then the α-limit set is

$$\alpha(x) = \{ y \in \mathbb{R}^n : \exists \text{ a sequence } (n_j) \text{ with } n_j \to \infty \text{ and } f^{n_j}(x) \to y \text{ as } j \to -\infty \}.$$

If $\omega(x) = \{x, f(x), f^2(x), \dots\}$
then $\alpha(x) = \{x, f^{-1}(x), f^{-2}(x), \dots\}$

A set M is <u>invariant</u> if $\forall x \in M$, $f^n(x) \in M$.

<u>Note</u>: limit sets are closed and invariant.

Cantor's middle-thirds set

Start with $0 \text{———} 1$. $l=1$

Take out any number that can't be written in base 3 with the digit 1. (each decimal place in term)

Do it for each position:

0 ——— $\frac{1}{3}$ $\frac{2}{3}$ ——— 1 $l = \frac{2}{3}$

0 — $\frac{1}{9}$ $\frac{2}{9}$ — $\frac{1}{3}$ $\frac{2}{3}$ — $\frac{7}{9}$ $\frac{8}{9}$ — 1 $l = \left(\frac{2}{3}\right)^2$

(i.e. take out middle third each line). 0 - -   - -   - -   - - 1 $l = \left(\frac{2}{3}\right)^3$

etc.

What is left at ∞ (eventually) is the Cantor middle third set C_∞.

<u>Five things about C_∞.</u>

1. C_∞ is a closed set as ~~a~~ union of closed sets
2. C_∞ is totally disconnected
3. C_∞ is perfect (i.e. has no isolated paths)
   and $\forall$ member $\exists$ an arbitrarily close other n°.
4. C_∞ has no length
5. C_∞ is uncountable

<u>Proofs</u>: 1. Obvious

2. Pick two n°s $0.a_1 a_2 \cdots a_n 0\, a_{n+1} a_{n+2} \cdots$

$0.a_1 a_2 \cdots a_n 2\, b_{n+1} b_{n+2} \cdots$

Consider $0.a_1 a_2 \cdots a_n 1\, 2\, c_{n+2}$, should go between them.
But this $\notin C_\infty$ $\because$ it has a 1.

4 ~~3~~. Let C_n be the set when n middle terms are removed. The length of $C_n = \left(\frac{2}{3}\right)^n$
$\Rightarrow$ length of $C_\infty = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$.

3. $\forall\, 0.a_1 \cdots a_n \cdots \in C_\infty$
$\exists\, 0.a_1 \cdots a_n (2-a_n)\, 2 \in C_\infty$

5. Two sets X and Y are countable if they have the same cardinality which means that $\exists$ an invertible map that pairs each $x \in X$ with each $y \in Y$.

$\mathbb{N}$ is countable, $[0,1]$ is uncountable

If $f: A \to B$ and B is count. then A is count.
If $f: C \to D$ and D is uncount. and f is surjective then C is uncount.

C_∞ has points base 3.
Let D_∞ be the set of points in $[0,1]$ base 2.
Let $f: C \to D$ be s.t. we're changing the 2s to 1s.

Note that $[0,1]$ base 2 is uncountable.
Show that cardinality of $[0,1]$ base 2 $\geq$ card $[0,1]$.

<u>OR</u>

list of n^os, countable.

Suppose $\{c_1, \dots, c_N, \dots\}$ be the full Cantor set.
Let c_{ij} denote the j^{th} digit in the expansion $0.a_1a_2\dots c_{ij}a_{j+1}$

Define $\bar{c} = \bar{c}_{11}\bar{c}_{22}$ where $\bar{\ }$ means swap 0s + 2s.

$\Rightarrow \bar{c} \in C_\infty$
$\Rightarrow \bar{c}$ not in the list $\{c_1, \dots c_N\}$ #

TOPOLOGICAL CONJUGACY AND CHAOS

Def^n: Let f and g be two maps s.t. $f: X \to X$ and $g: Y \to Y$.
f and g are topologically conjugate if there is a homeomorphism $h: X \to Y$ s.t. $h \circ f = g \circ h$.
h is then a "topological conjugacy".

Example: $f(x) = 4x(1-x)$ on $[0,1]$
$g(x) = 1 - 2x^2$ on $[-1,1]$

Show $\exists$ h s.t. $h \circ f = g \circ h$,

where $h(x) = 2x - 1$. $h \circ f = 2[4x(1-x)] - 1 = 8x - 8x^2 - 1$
$g \circ h = 1 - 2[2x-1]^2 = 8x - 8x^2 - 1$

Note: If f and g are conjugates, we can see that their second powers are conjugates with the same conjugacy, i.e. $h \circ f^2 = g^2 \circ h$.

Proof: $h \circ f \circ f = (h \circ f) \circ f = (g \circ h) \circ f$
$= g \circ (h \circ f) = g \circ (g \circ h) = g^2 \circ h$.

By induction, it is easy to show that conjugates are valid for the n^{th} power.
i.e. $h \circ f^n = g^n \circ h$.

Note: A map f has a fixed point x where $f(x) = x$.
If x is a period n point, then $f^n(x) = x$.

If x is a period n point of f then $h(x)$ is a period n point of g.

Proof: $f^n(x) = x \Rightarrow \underbrace{h \circ f^n(x) = h \circ x = h \Rightarrow g^n \circ h(x)}_{\text{equal}} = h(x)$

Reverse argument: If h is invertible and x is a period n point for g then $h^{-1}(x)$ is a period n point of f.

Exercise: Suppose f and g are conjugates, i.e. $h \circ f = g \circ h$. Suppose that the forward orbit of x under f converges to y, i.e. $\lim_{n\to\infty} f^n(x) = y$.
Show that the forward orbit of $h(x)$ converges to $h(y)$ under g, i.e. $\lim_{n\to\infty} g^n(h(x)) = h(y)$.

$$f^n(x) \to y \quad \text{as } n\to\infty$$
$$h\circ f^n(x) \to h(y) \text{ as } n\to\infty$$
$$g^n\circ h(x) \to h(y) \text{ as } n\to\infty.$$

Conjugacy $\Leftrightarrow$ maps are injective and surjective.
Semi-conjugacy $\Leftrightarrow$ maps are surjective but not necessarily injective.

Def^{n}: A map $f: M\to M$ is said to be topologically transitive if for any pair of open sets $U, V \subseteq M$, there exists $k>0$ s.t. $f^k(U)\cap V \neq \emptyset$.

Recall the open sets U, V are s.t. $\exists\, n$ s.t. $f^n(x)\in U$ and $\exists\, m$ s.t. $f^m(x)\in V$. thus for every U and V, some points of U are mapped onto V.

Def^{n}: Sensitive dependence on initial conditions.

A map $f: M\to M$ has sensitive dependence on initial conditions if $\exists\, \delta>0$ s.t. $\forall x\in M$ and any neighbourhood N of x, $\exists\, y\in N$ and $n\geq 0$ s.t.
$$|f^n(x)-f^n(y)| > \delta$$

Example: $f(\theta) = \theta + 2\pi r$, $0<r<1$, r irrational
Every point has a dense orbit $\Rightarrow$ the map is T.T.

An irrational rotation of the circle is topologically transitive but does not have sensitive dependence on initial conditions

KP ON MONDAY

2. Semi-conjugacy

This is a weaker property than conjugacy. If we have a map h, s.t. $h \circ f = g \circ h$, where h is continuous onto, but not necessarily one-to-one, we have semiconjugacy between f and g.

This means that $h \circ f^n = g^n \circ h$.
Thus if f^n has a fixed point x, we get

$$f^n(x) = x \Rightarrow h \circ f(x) = h(x)$$
$$\Rightarrow g^n(h(x)) = h(x)$$

$h[f(x)] = h[x]$

ie. $h(x)$ is a fixed point of g^n.

This time, however, it is not necessarily of prime period n when x is of prime period n under f, since the map is not one-to-one.

Thus if f is a map we know about and we have a semi-conjugacy with a map g of the form $h \circ f = g \circ h$, then this tells us some things about g, but not everything.

To see why we are interested in conjugacy and semi-conjugacy we now turn to ideas connected with chaos.

Example of a semiconjugacy

$$\begin{tikzcd}
S^1 \arrow[r, "2\theta"] \arrow[r, swap, "\text{doubling map}"] \arrow[d, swap, "h \to -\cos\theta"] & S^1 \arrow[d, "-\cos\theta \leftarrow h"] \\
{[-1,1]} \arrow[r, swap, "1-2x^2"] & {[-1,1]}
\end{tikzcd}$$

$h: S^1 \to [-1, 1]$ $\qquad h(\theta) = -\cos\theta$

h is continuous and onto
but not one-to-one.

So h is a semiconjugacy $\qquad$ (e.g. $h\left(\frac{\pi}{2}\right) = h\left(\frac{3\pi}{2}\right)$)

3. Definitions of chaos

If M is an invariant set for a map, we can treat the map on M, i.e. $f|_M$ as a separate map. For simplicity we will write $f: M \to M$ for the restriction of f to M.

In the following definitions, f is some invariant subset of X.

Intuitively, a map is <u>topologically transitive</u>, if, given any neighbourhood, some points in this neighbourhood move under iteration to any other neighbourhood.

If a map has a <u>dense orbit</u> then it is topologically transitive.

Consider a point x in the dense orbit. Given any open sets U and V, by definition there is some n s.t. $f^n(x) \in U$ and some $m > n$ s.t. $f^m(x) \in V$. Thus for every U and V, some points in U are eventually mapped into V.

*

<u>Def^n</u>: <u>Chaotic</u>: (many similar definitions)

Let M be a set. $f: M \to M$ is <u>chaotic</u> if

1) f has SDIC
2) f is TT
3) periodic points are dense in M (DPP)

<u>Example</u>: <u>Doubling map is chaotic</u>:

We are interested in $T: [0,1) \to [0,1)$ defined by $T(x) = 2x \bmod 1$ (or equivalently the doubling map on the circle $f: S^1 \to S^1$ given by $f(\theta) = 2\theta$). We know that periodic points ($\mathbb{Q}$) are dense. We can prove that T is TT because T has a dense orbit.

<u>Exercise</u>: Construct a point which has a dense orbit.

It is easy to see that f has SDIC because every distance is doubled, so nearby points eventually separate. Alternatively, consider any $x = 0.x_0 x_1 x_2 \ldots$ written in binary. We can find a point y arbitrarily close to x which eventually separates from x by a distance at least $\frac{1}{2}$ as follows:

Let y differ from x only in the nth place, e.g.

$$x = x_0 x_1 \ldots x_{n-1} 0 x_{n+1} x_{n+2} \ldots \qquad y = x_0 x_1 \ldots x_{n-1} 1 x_{n+1} x_{n+2} \ldots$$

You can check $|f^n(x) - f^n(y)| = \frac{1}{2}$.

But since n was arbitrary, x and y cannot be as close as we choose.

<u>Exercise</u>: Convince yourself that numbers of the form $\frac{k}{2^n}$ are dense in $\mathbb{R}$.

dyadic rationals

4. Conjugacy, semiconjugacy and chaos

Assume we have two maps $f: X \to X$ and $g: Y \to Y$ and there is a semi-conjugacy $h \circ f = g \circ h$, between the two. One way of expressing this is to say that this diagram commutes:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

We prove that some aspects of chaos are preserved by semi-conjugacy. For the arguments below we need to note the following about the set-theoretic inverse:

Consider a set U and its inverse $h^{-1}(U)$.
Clearly $h(h^{-1}(U)) = U$, since h is onto.

<u>Lemma</u>: If periodic points of f are dense in X, then periodic points of g are dense in Y.
(Semiconjugacy preserves dense periodic pts).

<u>Proof</u>: Consider any point $y \in Y$ and some neighbourhood U of y. Since h is continuous and onto, $V \equiv h^{-1}(U)$ is a nonempty open set in X.

Since periodic points are dense in X, $\exists$ a periodic point $p \in V$. But then $h(p) \in U$ is a periodic point of g. Since y and U are arbitrary, this proves that periodic points of g are dense in Y.

<u>Lemma</u>: If f is TT, so is g. (semi-conjugacy preserves TT)

<u>Proof</u>: Take two open sets U and V in Y. Since h is continuous and onto, the sets $h^{-1}(U)$ and $h^{-1}(V)$ are nonempty open sets in X.

Since f is TT, $\exists\ k > 0$ s.t. $f^k[h^{-1}(U)] \cap h^{-1}(V) \neq \emptyset$

i.e. $h \circ f^k[h^{-1}(U)] \cap V \neq \emptyset$

i.e. $g^k \circ h\ [h^{-1}(U)] \cap V \neq \emptyset$.

$$g^k(U) \cap V \neq \emptyset. \qquad \square$$

We cannot prove (because it isn't true) that semi-conjugacy preserves SDIC (but conjugacy <u>does</u>.)

<u>Lemma</u>: If h is a conjugacy s.t. h^{-1} exists and is continuous, then if f has SDIC, then so does g.

<u>Proof</u>: Consider a point $y \in Y$, and a neighbourhood U of y. Choose some point $x \in X$ which maps to Y, i.e. $h(x) = y$, since h is continuous, $h^{-1}(U)$ will contain some neighbourhood V of x.

Since f has SDIC, there is some point $p \in V$, some $n \geq 0$ and some $\delta > 0$ s.t.

$$|f^n(x) - f^n(p)| > \delta.$$

Define $q = h(p)$, so we can write
$$|f^n(h^{-1}(y)) - f^n(h^{-1}(q))| > \delta$$

which we can write as
$$|h^{-1}(g^n(y)) - h^{-1}(g^n(q))| > \delta$$

$(h \circ f = g \circ h \Rightarrow f \circ h^{-1} = h^{-1} \circ g)$

Since h^{-1} is continuous, this means $\exists\, \varepsilon > 0$ s.t.
$|g^n(y) - g^n(q)| > \varepsilon \implies g$ has SDIC.

We have shown that all the ingredients of chaos are preserved by conjugacy. Also 2 of 3 are preserved by semi-conjugacy. The third (SDIC) may or may not be preserved, but in many applications (see below) it is.

This gives us one way of showing that a map is chaotic.

Exercise: Give an example of a (trivial) semi-conjugacy which does not preserve sensitive dependence on initial condtns.

Example: The map $f(x) = 4x(1-x)$ on $[0,1]$.
Earlier we saw that $f(x)$ is conjugate via a linear conjugacy to $F(x) = 1-2x^2$ on $[-1,1]$.

We now show that $F(x)$ is chaotic on $[-1,1]$.
This will imply that $f(x)$ is chaotic on $[0,1]$.

Define $h: S^1 \to [-1,1]$ by $h(\theta) = -\cos\theta$.
h is continuous and onto but not one-to-one.
It is $\therefore$ a semiconjugacy but not a conjugacy.

Now consider the doubling map on the circle, $g(\theta) = 2\theta$, which we have studied and proved is chaotic.

We can check that

$$h \circ g(\theta) = -\cos(2\theta) = 1 - 2\cos^2\theta = F \circ h(\theta)$$

$$\begin{tikzcd}
S^1 \arrow[r, "h(\theta) = -\cos\theta"] \arrow[d, "g(\theta)=2\theta"'] & {[-1,1]} \arrow[d, "F(\theta) = 1-2\theta^2"] \\
S^1 \arrow[r, "h(\theta) = -\cos\theta"'] & {[-1,1]}
\end{tikzcd}$$

Thus h is a semi-conjugacy between $g(\theta) = 2\theta$ and $F(x) = 1 - 2x^2$.

This tells us immediately that

1) Periodic points of F are dense in $[-1, 1]$
2) F is TT.

It remains to show that F has SDIC.

Given any $x \in [-1, 1]$ and some neighbourhood U of x, we can find a $y \in S^1$ and a neighbourhood V of y satisfying $h(y) = x$ and $h(V) = U$ (because h is continuous and onto).

If the length of V is ε, then the length of $g^n(V)$ is $\min\{2^n\varepsilon, 2\pi\}$, where g is the doubling map.

$\Rightarrow$ for n sufficiently large, $g^n(V) = S^1$.

$$\Rightarrow \quad \underset{\substack{\parallel \\ F^n \circ h(V) \\ \parallel \\ F^n(U)}}{h \circ g^n(V)} = [-1, 1] \qquad \Rightarrow F \text{ has SDIC.}$$

$\Rightarrow$ $F(x) = 1-2x^2$ and hence $f(x) = 4x(1-x)$ are chaotic on $[-1,1]$ and $[0,1]$ respectively.

Note that without semi-conjugacy with the doubling map, this would have been very hard to prove.

Logistic family; higher dimensional maps

Taylor expansion: As long as a f^n $f: \mathbb{R} \to \mathbb{R}$ is sufficiently differentiable, it can be expanded in a Taylor series.

eg. $$f(x+\delta) = f(x) + \delta f'(x) + \frac{\delta^2}{2} f''(x) + O(\delta^3)$$

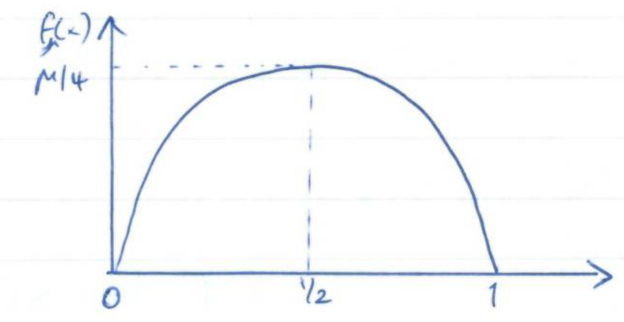
Exploring the logistic family

The logistic family of maps is defined as
$$x_{n+1} = f_\mu(x_n) = \mu x_n (1-x_n)$$

μ is a parameter which can be varied.

We are interested in the dynamics of points in $[0,1]$ for $\mu > 0$

The maps have one critical point at $x = \frac{1}{2}$, and as μ increases, $f_\mu(\frac{1}{2}) = \frac{\mu}{4}$ increases.



What type of point is 0? It's a fixed point since $f_\mu(0) = 0$.

What type of point is 1? It maps onto 0 so it's an eventually fixed point. ie. $f_\mu^n(1) = 0 \;\forall n \geq 1$

<u>Fixed points</u>: $x_* = f_\mu(x_*) = \mu x_*(1-x_*)$

$$\Rightarrow x_* = 0 \quad \text{or} \quad 1 = \mu(1-x_*)$$
$$x_* = \frac{\mu-1}{\mu}.$$

We are interested in points in $[0,1]$ $\Rightarrow$ for $\mu \leq 1$, there is one fixed point at $x_* = 0$. For $\mu > 1$, there are 2 fixed points at $x_* = 0$ and $(\mu-1)/\mu$.

<u>Stability of fixed points</u>:
$$f_\mu'(x) = \mu(1-2x)$$
$$f_\mu'(0) = \mu$$
$$f_\mu'\left(\frac{\mu-1}{\mu}\right) = 2-\mu$$

So $|f_\mu'(0)| < 1$ iff $\mu < 1$
$\quad |f_\mu'(\frac{\mu-1}{\mu})| < 1$ iff $1 > \mu > 3$

$\mu < 1$: $\exists$ a stable fixed pt at 0 (global attractor, actually)
$1 > \mu > 3$: $\exists$ a repelling fixed pt at 0
an attracting fixed pt at $\frac{\mu-1}{\mu}$.
$\mu = 1$: The fixed point at zero loses stability in a "bifurcation" (week 10).
$\mu = 3$: The fixed point at $(\mu-1)/\mu$ loses stability in a bifurcation.

Period-2 orbits

$$x = f_\mu^2(x) \quad \leftarrow \text{looking for fixed points of } f_\mu^2.$$
$$= \mu(\mu x(1-x))(1-\mu x(1-x))$$
$$= \mu^2 x(1-x)(1-\mu x(1-x))$$

either $\underline{x=0}$ or $\frac{1}{\mu^3} = (x^2 - x + \frac{1}{\mu})(1-x)$

$$\Rightarrow x^3 - x^2 + \frac{x}{\mu} - x^2 + x - \frac{1}{\mu} + \frac{1}{\mu^3} = 0$$

$$\Rightarrow x^3 - 2x^2 + x\left[1 + \frac{1}{\mu}\right] - \frac{1}{\mu} + \frac{1}{\mu^3} = 0 \quad \text{or } x = 0$$

$$\Rightarrow \left(x - \left(\frac{\mu-1}{\mu}\right)\right)\left(x^2 - x\left(1+\frac{1}{\mu}\right) + \frac{1}{\mu}\left(1+\frac{1}{\mu}\right)\right) = 0$$

$x = \frac{\mu-1}{\mu}$, fixed pt

Period 2 cycle consists of the roots of
$$x^2 - x\left(1+\frac{1}{\mu}\right) + \frac{1}{\mu}\left(1+\frac{1}{\mu}\right) = 0$$

Roots not real if $\left(1+\frac{1}{\mu}\right)^2 - 4\frac{1}{\mu}\left(1+\frac{1}{\mu}\right) < 0$

$$\Leftrightarrow 1 + \frac{1}{\mu} < \frac{4}{\mu} \Leftrightarrow 1 < \frac{3}{\mu} \Leftrightarrow \mu < 3$$

$\Rightarrow$ Period 2 cycle exists for $\mu > 3$

Call the 2 points in a period-2 cycle x_1 and x_2.

$$(f_\mu^2)'(x_i) = f_\mu'(x_1) f_\mu'(x_2) \quad \ldots \quad \text{for } i = 1, 2$$
$$= \mu(1-2x_1)\mu(1-2x_2)$$
$$= \mu^2(1 - 2(x_1+x_2) + 4x_1x_2)$$

x_1, x_2 are roots of $x^2 - x\left(1+\frac{1}{\mu}\right) + \frac{1}{\mu}\left(1+\frac{1}{\mu}\right) = 0$

$$(x-x_1)(x-x_2) \qquad x_1+x_2 = 1+\frac{1}{\mu} \qquad x_1x_2 = \frac{1}{\mu}\left(1+\frac{1}{\mu}\right)$$

$$\begin{aligned}(f_\mu^{2})'(x_i) &= \mu^2\left[1-2\left(1+\frac{1}{\mu}\right)+\frac{4}{\mu}\left(1+\frac{1}{\mu}\right)\right] \\ &= -\mu^2+2\mu+4 \\ &= -(\mu-1)^2+5\end{aligned}$$

2-cycle is stable $\iff |(f_\mu^{2})'(x_i)| < 1$

$$\begin{aligned}&\iff |-(\mu-1)^2+5| < 1 \\ &\iff 4 < (\mu-1)^2 < 6 \\ &\iff 3 < \mu < 1+\sqrt{6}\end{aligned}$$

$\mu=1,\ \mu=3,\ \mu=1+\sqrt{6}$

$3 \leq \mu < 4$: There is a cascade of bifurcations leading the creation of a period-2 orbit, then a period-4 orbit, then a period-8 orbit, etc.

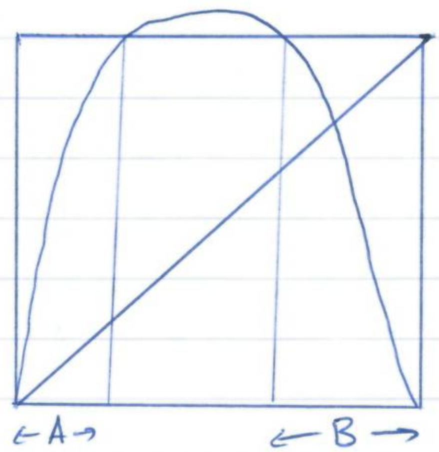
After this, there are regions where there is a stable periodic orbit, and regions where there is no stable periodic orbit. $\Rightarrow$ complicated behaviour.

For $\mu=4$, the map is semi-conjugate to the doubling map, plus one other thing $\Rightarrow$ chaotic.

Mathematica task:

- Select initial $x_0 \in [0,1]$.
- Plot x_n against n for $n = 0, \ldots,$ with
  a) $\mu = 0.8$
  b) $\mu = 2.8$
  c) $\mu = 3.3$
  d) $\mu = 3.5$
  e) $\mu = 3.9$.

For $\mu > 4$



$$A \to A \cup B$$
$$B \to A \cup B$$

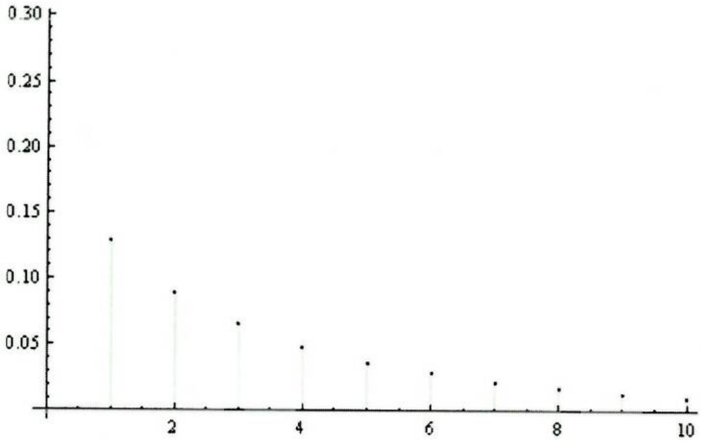
There is a Cantor set of points in $A \cup B$, which are never mapped out of the unit interval. The dynamics on the Cantor set is chaotic. We do not prove it, but it is easy to prove if μ is large enough.

Finding stable periodic orbits in the logistic family

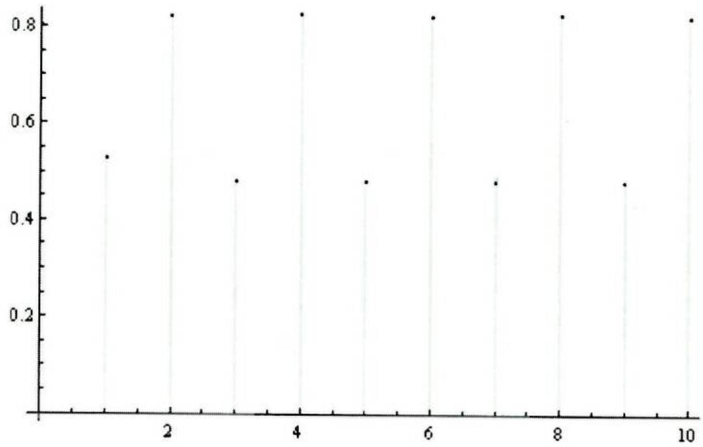
It is possible to use the IVT to find regions of μ where there are stable periodic orbits. The theorem we will prove is one example.

$x_0 = 0.8$

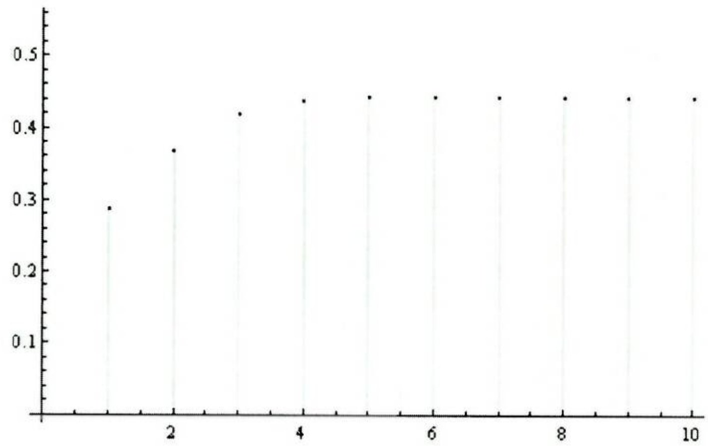
$\mu = 0.8$



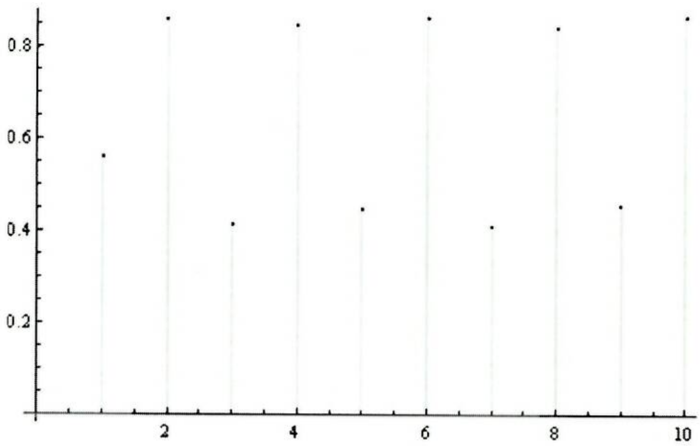
$\mu = 3.3$



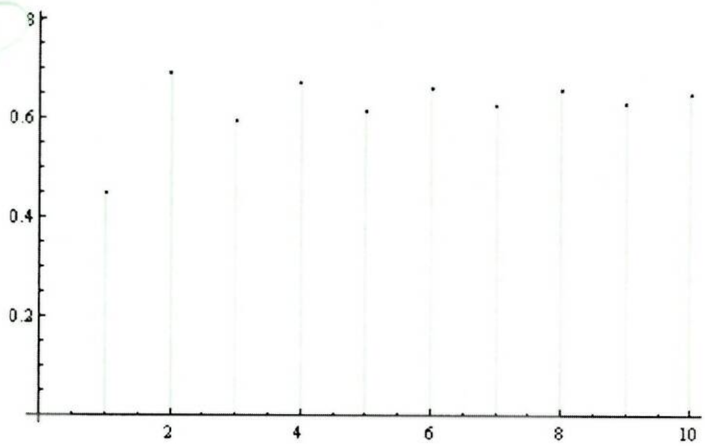
$\mu = 1.8$



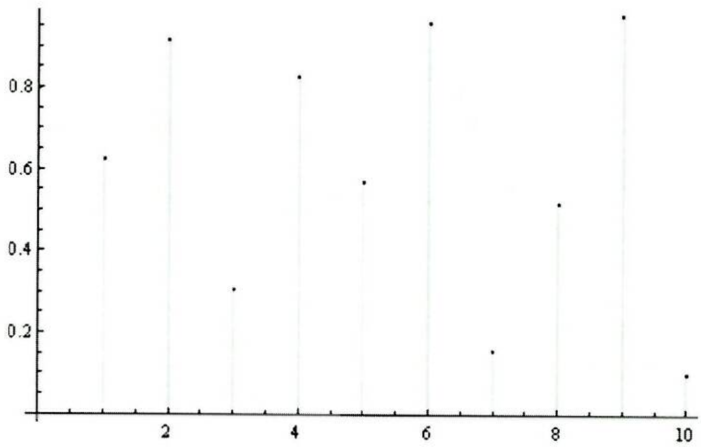
$\mu = 3.5$



$\mu = 2.8$



$\mu = 3.9$



$f(x) = \mu x(1 - x)$

$x_n = f(x_{n-1})$

Note that it describes a very different situation from previous results where we found periodic orbits of all periods for one map.

Those were (in general) unstable and existed for one map. The periodic orbits we will now construct are stable, and only one exists at any one parameter value.

<u>Defn</u>: <u>superstable</u> : A period n point p of a map f is <u>superstable</u> if $(f^n)'(p) = 0$. This means that some iterate of p falls on the critical point of the map. (by the chain rule thing about $(f^n)'(p)$).

Question: when is $\frac{\mu-1}{\mu}$ superstable?

$$f_\mu'\left(\frac{\mu-1}{\mu}\right) = 0$$

$$\mu\left(1-2\left(\frac{\mu-1}{\mu}\right)\right)=0$$

$$2-\mu=0$$

$$\mu=2$$

or $f'(x)=0$ at $x=\frac{1}{2}$ on logistic map

$$\Rightarrow \frac{\mu-1}{\mu} = \frac{1}{2}$$

$$2\mu-2=\mu$$

$$\mu=2.$$

<u>Theorem</u>: Given any $n \geq 1$, there is a value of $\mu \in [2,4)$ s.t. f_μ has a superstable periodic orbit of prime period n.

This theorem implies that there are many regions where the logistic map has some stable periodic behaviour. This does not mean that there is no chaotic behaviour at these parameter values, just that if there <u>is</u>

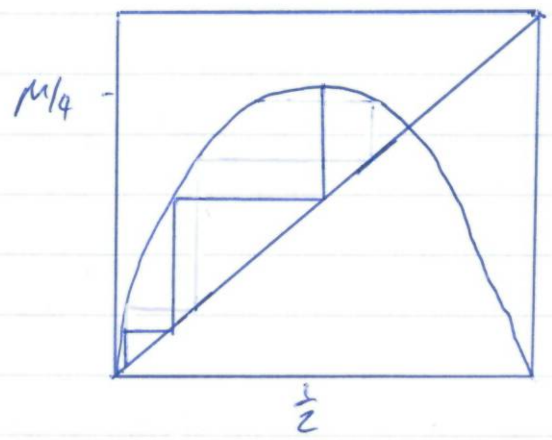
chaotic behaviour, then it coexists with stable periodic behaviour.

Sketch proof: For $n = 1, 2$, we can explicitly do the calculation to find the values of μ where there are superstable orbits. To prove the theorem for $n \geq 3$, we first need:

Lemma: For any $n \geq 1$ and any $\mu \in (2, 4)$, there is a point $x_{n,\mu} \in (0, \frac{1}{2})$ s.t. $f_\mu^n(x_{n,\mu}) = \frac{1}{2}$.

In other words, the turning point $\frac{1}{2}$ has preimages in $(0, \frac{1}{2})$ for every n.

These preimages become closer and closer to 0 as n becomes large.



Proof: (drop the subscript μ)

$n = 1$: $f(\frac{1}{2}) = \frac{\mu}{4} > \frac{1}{2}$
$f(0) = 0 < \frac{1}{2}$
By IVT, $\exists\, x \in (0, \frac{1}{2})$ s.t. $f(x_1) = \frac{1}{2}$.

Let $x_n \in (0, \frac{1}{2})$
$f^n(x_n) = \frac{1}{2}$.

$f^{n+1}(x_n) = f(\frac{1}{2}) = \frac{\mu}{4} > \frac{1}{2}$.
$f^{n+1}(0) = 0$.

So $\exists\, x_{n+1} \in (0, x_n)$ s.t. $f^{n+1}(x_{n+1}) = \frac{1}{2}$

Lemma $\square$.

We now fix n and allow μ to vary.

We apply the IVT again, noting that f_μ is continuous in both x and μ and that $x_{n,\mu}$ is a continuous function of μ. (in order to prove this rigorously, we need the implicit value theorem).

We can check directly that $f^2_{2.5}(\frac{1}{2}) > \frac{1}{2}$

and $f^2_4(\frac{1}{2}) = 0$. So $f^2_{2.5}(\frac{1}{2}) - x_{n,2.5} > 0$

and $f^2_4(\frac{1}{2}) - x_{n,4} < 0$.

So there is $\mu_n \in (2.5, 4)$ s.t. $f^2_{\mu_n}(\frac{1}{2}) = x_{n,\mu_n}$.

But then $f^{n+2}_{\mu_n}(\frac{1}{2}) = f^n_{\mu_n}(x_{n,\mu_n}) = \frac{1}{2}$.

Thus $\frac{1}{2}$ is a periodic point of period $n+2$ for the map f_{μ_n}.

We can easily check that by construction, $n+2$ is the prime period of x_{n,μ_n}.

Of course, this point is superstable since $f'(\frac{1}{2}) = 0$.

If a map has a stable period n orbit at a parameter value μ_n, then there is an open interval of M around μ_n for which the map has a stable period n orbit. Thus the above theorem implies that there is an open parameter set at which the family f_μ has stable periodic orbits.

Look at $0<\mu<1$.

Claim: 0 is globally stable.

Proof: If $x \in [0,1]$, $f_\mu(x) = \mu x(1-x)$

$$0 \leq f_\mu(x) \leq \mu x$$

$$0 \leq f_\mu^n(x) \leq \mu f_\mu^{n-1}(x) \leq \mu^2 f_\mu^{n-2}(x) \leq \cdots \leq \mu^n x$$

$\forall x \in [0,1]$ as $n \to \infty$, $f_\mu^n(x) \to 0$, since $0<\mu<1$

$\Rightarrow$ 0 is globally stable. $\square$

Look at $1<\mu<2$.

Claim: $\frac{\mu-1}{\mu}$ is globally stable on $(0,1)$

f is strictly increasing on $[0,\frac{1}{2})$ and $f\left(\frac{\mu-1}{\mu}\right) = \frac{\mu-1}{\mu}$, $f(0) = 0$

$$\Rightarrow \quad 0 \leq f(x) \leq \frac{\mu-1}{\mu} \quad \text{for } x \in \left[0, \frac{\mu-1}{\mu}\right] \quad \cdots (i)$$

$\mu(1-x) \geq 1$ for $x \in \left[0, \frac{\mu-1}{\mu}\right] \Rightarrow f(x) \geq x$ for $x \in \left[0, \frac{\mu-1}{\mu}\right]$ $\cdots$ (ii)

(i) and (ii) imply that the sequence $f^n(x)$ is increasing for $x \in [0, \frac{\mu-1}{\mu}]$. It is also bounded above by $\frac{\mu-1}{\mu}$.

$\Rightarrow f^n(x)$ converges to a limit $\geqslant x$.

But we know that if an orbit tends to a limit, that limit must be a fixed point.

$\Rightarrow f^n(x) \to 0$ or $\frac{\mu-1}{\mu}$, but limit must be $> x$.

$\forall x \in [0, \frac{\mu-1}{\mu}]$, $f^n(x) \to \frac{\mu-1}{\mu}$.

Similarly on $[\frac{\mu-1}{\mu}, \frac{1}{2}]$, $f(x) \leqslant x$ and $f(x) \in [\frac{\mu-1}{\mu}, \frac{1}{2}]$.

So $f^n(x)$ is a decreasing sequence bounded below by $\frac{\mu-1}{\mu}$.

$\Rightarrow f^n(x)$ tends to a limit $\geqslant \frac{\mu-1}{\mu}$.

Since this limit must be a fixed point of f,

$$f^n(x) \to \frac{\mu-1}{\mu} \qquad \forall x \in (\tfrac{\mu-1}{\mu}, \tfrac{1}{2}),$$

$\Rightarrow f^n(x) \to \frac{\mu-1}{\mu} \quad \forall x \in (0, \frac{1}{2})$

Now $x \in (0, 1) \Rightarrow f(x) \in (0, \frac{\mu}{4}) \subseteq (0, \frac{1}{2})$.

$\Rightarrow \forall x \in (0,1) \Rightarrow f^n(x) \to \frac{\mu-1}{\mu}$ as $n \to \infty$ $\square$.

Look at $2<\mu<3$ $\frac{\mu-1}{\mu}$ is globally stable (a bit harder to prove).

$\mu=3$ is non-hyperbolic — $\frac{\mu-1}{\mu}$.

End of logistic map.

A quick look at higher dimensional maps

The 1D maps that we have studied which displayed complicated behaviour were all non-invertible.

Continuous, invertible maps in 1D are forced to be monotonic, and thus means that they only can display very simple behaviour.

But in higher dimensions this is not true: invertible maps can display very complicated behavior, including chaotic behaviour. Maps in higher dimensions can be studied using some of the same techniques as maps in 1D, but some of the techniques can no longer be applied (e.g. IVT). Also results on the implications of a period 3 orbit relied heavily on the IVT, so they are only true in 1D.

Finding fixed points

A general map on $\mathbb{R}^2$ takes the form:

$$x_{n+1} = f(x_n, y_n)$$
$$y_{n+1} = g(x_n, y_n) \,.$$

Fixed points occur at solutions of the simultaneous eq^{n}s

$$x = f(x,y) \qquad\qquad y = g(x,y)$$

Each of these eq^{n}s is a scalar eq^n in $\mathbb{R}^2$, and in general defines a curve. Simultaneous solutions to both eq^{n}s are pts where the two curves intersect.

Of course the curves could be very complicated, and each curve may consist of several components.

For example, if $f(x,y) = x(x^2-y+1)$, then the eq^n $x = f(x,y)$ is solved by both $x=0$ and $y=x^2$

Example: Consider the 2D map

$$x_{n+1} = x_n^2 - y_n$$
$$y_{n+1} = y_n^2 + x_n y_n$$

Fixed pts:

$$x = x^2 - y$$
$$y = y^2 + xy \Rightarrow y=0 \quad \text{or} \quad y = 1-x$$

$$y=0 \Downarrow x=0,1 \Downarrow x=0,\ y=0$$

$$y=1-x \Downarrow x=\pm 1 \Downarrow x=1,\ y=0; \quad x=-1,\ y=2$$

Fixed pts: $(0,0)$, $(1,0)$ and $(-1,2)$

Example: Consider the map

$$x_{n+1} = 2x_n$$
$$y_{n+1} = \tfrac{1}{4} y_n$$

What are the fixed points? $(0,0)$.

What is the long-term behaviour

$$x_n = 2^n x_0 \Rightarrow x_n \to \infty \text{ as } n \to \infty$$
$$y_n = \left(\tfrac{1}{4}\right)^n y_0 \Rightarrow y_n \to 0 \text{ as } n \to \infty$$

so $(0,0)$ is unstable.

Example: Consider the map
$$x_{n+1} = 2y_n$$
$$y_{n+1} = \tfrac{1}{4} x_n$$

What are the fixed pts? $(0,0)$

What is the long-term behaviour? Stable.

$$y_{n+1} = \tfrac{1}{4} x_n = \tfrac{1}{2} y_{n-1}$$
$$\Rightarrow y_{2n} = \left(\tfrac{1}{2}\right)^n y_0$$
$$y_{2n+1} = \left(\tfrac{1}{2}\right)^n y_1 = \left(\tfrac{1}{2}\right)^n \tfrac{1}{4} x_0.$$

$$\Rightarrow y_n \to 0 \text{ as } n \to \infty.$$

Similarly $x_{n+2} = 2y_{n+1} = \tfrac{1}{2} x_n$

$$\Rightarrow x_{2n} = \left(\tfrac{1}{2}\right)^n x_0$$

$$x_{2n+1} = \left(\tfrac{1}{2}\right)^n x_1 = \left(\tfrac{1}{2}\right)^n 2y_0$$

$$\Rightarrow x_n \to 0 \text{ as } n \to \infty$$

$\Rightarrow$ Stable.

Let us look for a more systematic method of analysis.

Stability

Consider the map $x_{n+1} = 2y_n$, $y_{n+1} = -\frac{1}{3}x_n$ and follow the orbit of the point $(1,1)$.

$$(1,1) \to (2, -\tfrac{1}{3}) \to (-\tfrac{2}{3}, -\tfrac{2}{3}) \to (-\tfrac{4}{3}, \tfrac{2}{9}) \to (\tfrac{4}{9}, \tfrac{4}{9}) \to \dots$$

If we plot these points we see they're spiralling towards $(0,0)$. This is a "linear" map, so let's write it in matrix form as

$$\underline{x}_{n+1} = A\underline{x}_n \quad \text{where} \quad \underline{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 2 \\ -\frac{1}{3} & 0 \end{pmatrix}.$$

Thm: local stability of fixed pts

Consider a C^1 map $f(\underline{x})$, where $\underline{x} \in \mathbb{R}^n$. A fixed point p is locally stable iff all eigenvalues of the Jacobian (ie. matrix of 1^{st} partial derivatives) of the map evaluated at the fixed point – ie. of $\underline{D}f(\underline{p})$ – lie inside the unit circle. A period n point p is stable if all eigenvalues of $\underbrace{\underline{D}f^n(p)}_{\text{Jacobian of } f^n}$ lie inside the unit circle.

Example: Find the fixed points of the following 2D system and calculate their stability:

$$x_{n+1} = 3x_n + y_n^2 \qquad y_{n+1} = 2y_n - x_n$$

To find fixed points, $x = 3x + y^2$ $\quad 0 = 2x + x^2$
$y = 2y - x \Rightarrow x = y$ $\quad \Rightarrow x = 0, -2.$

$$J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} 3 & 2y \\ -1 & 2 \end{pmatrix}$$

$$J\big|_{(0,0)} = \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} \Rightarrow \text{e'vals are } 3, 2.$$

$$J\big|_{(-2,-2)} = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix} \Rightarrow \text{e'vals:}$$

$$(3-\lambda)(2-\lambda) - 4 = 0$$
$$\lambda^2 - 5\lambda + 2 = 0$$
$$\rightarrow \lambda = \frac{5 \pm \sqrt{17}}{2}$$

$(-2,-2)$ is a fixed point which is unstable.
(a saddle type, actually)
$(0,0)$ is also unstable.

Hyperbolicity of fixed points in higher dimensional maps

In the 1D case, we saw that a fixed point was hyperbolic as long as the derivative of the map, evaluated at the fixed point, did not have absolute value 1. The general def^n is:

Def^n: hyperbolic : A fixed pt p of a C^1 map is hyperbolic if none of the eigenvalues of $Df(p)$ (ie. the Jacobian evaluated at the fixed pt) lie on the unit circle.

If, on the other hand, there are any eigenvalues with modulus 1, then the fixed pt is nonhyperbolic.

We see that as before, hyperbolicity is about the <u>linear part of the map</u>. If a fixed point is hyperbolic, this implies that it is possible to decide whether it is stable or unstable (and essentially say all there is to say about the structures of orbits near the fixed point) by looking only at the linear part of the map.

As in the 1D case, nonhyperbolic fixed points are associated with bifurcations, to be discussed in week 10.

Chaos in higher dimensional maps

One very famous 2D map is the Hénon map. It is a map on $\mathbb{R}^2$ defined by the equations

$$x_{n+1} = y_n + 1 - \alpha x_n^2$$
$$y_{n+1} = \beta x_n \qquad \alpha, \beta \text{ parameters.}$$

For many values of the parameters, initial conditions converge to an object of this form:

Blow-ups of small regions of the attractor reveal that what appear to be 1D curves have further internal structure.

$\alpha = 1.4$
$\beta = 0.3$

Moreover, the dynamics on the attractor is chaotic. This means that although initial conditions converge to the attractor, nearby initial conditions on the attractor separate (SDIC). Also there are many dense orbits (topological transitivity) and many periodic points.

Exercise: Confirm that the Hénon map is, in general, continuous and invertible.
Find any values of α and β for which it is not invertible.

The Cantor Middle Thirds Set

Properties

i) C_∞ is closed

Proof It is the intersection of closed sets

ii) C_∞ is totally disconnected. (This means it contains no intervals)

Proof

Consider 2 general elements of C_∞ (where $n+1$ is the decimal place at which they first differ) in base 3:

$$0.a_1 a_2 \cdots a_n 0 a_{n+1} a_{n+2} \cdots$$
$$0.a_1 a_2 \cdots a_n 2 b_{n+1} b_{n+2} \cdots ,$$

C_∞ does not contain $0.a_1 a_2 \cdots a_n 1 2$ which is between them. $\therefore$ for 2 general elements C_∞ does not contain the interval between them, i.e. C_∞ does not contain any intervals

iii) C_∞ is perfect. (This means it has no isolated points)

Proof Consider $0.a_1 a_2 \cdots a_n \cdots \in C_\infty$ (base 3)

$$0.a_1 a_2 \cdots a_{n-1}(2-a_n)2 \in C_\infty$$

and is arbitrarily close to $0.a_1 \cdots a_n \cdots$

iv) C_∞ has no length.

Proof Consider C_n = the set when n middle thirds have been removed (e.g. $C_1 = [0,1/3] \cup [\frac{2}{3},1]$, $C_0 = [0,1]$). The length of C_n is $\left(\frac{2}{3}\right)^n$ (by induction). $\therefore$ as $n \to \infty$, the length of $C_n \to 0$. $\therefore$ C_∞ has zero length.

v) C_∞ is uncountable

Proof We can define a surjective map from C_∞ base 3 onto $[0,1]$ base 2, by changing 2s to 1s. This means the cardinality of C_∞ is no less than $[0,1]$, which is uncountable $\square$

CONTINUOUS TIME SYSTEMS

This part of the course is concerned with understanding ODEs. We will look at ODEs as continuous-time dynamical systems, and our emphasis will be on understanding how they behave geometrically.

What is an ODE?

An ODE is an eqⁿ in one independent variable, some dependent variables, and the derivatives of the dependent variables w.r.t. the independent variable. If we call the dependent variables x and the independent variable t ("time"), we get:

$$0 = f\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^nx}{dx^n}, t\right).$$

n, the highest derivative of x, is the order of the ODE. For convenience, we'll often write $\frac{dx}{dt}$ as $\dot{x}$ and $\frac{d^2x}{dt^2}$ as $\ddot{x}$ etc.

Example: An example of an ODE is

$$\sin\left(\frac{d^2x}{dt^2} + \frac{dx}{dt}\right) + \frac{d^3x}{dt^3} = t\cos x \qquad \ldots (*).$$

It has order 3.

Rather than study ODEs in this general form, we'll look at ODEs of the form $\dot{x} = f(x,t)$.
("standard form")

It looks quite different at first sight: it's first order, ie. it only has first derivatives of x, and all the derivatives occur on the LHS.

Below we'll see that we can always reduce higher order ODEs to first order ones. But it is not always true that we can solve for the derivatives, and take them over to the LHS of the eq^n.

If the RHS of the eq^n depends explicitly on time, then the ODE is said to be nonautonomous. Otherwise it's autonomous, i.e.

$$\dot{x} = f(x) \qquad \qquad \dot{x} = f(x,t)$$

AUTONOMOUS NONAUTONOMOUS

Note: we can always reduce the system to 1st order, but we can't always put it in standard form.

Example: $x^2 + \dot{x}^2 = 4$

Can reduce to 1st order:

$$y = \dot{x}$$

$$\Rightarrow \quad x^2 + y^2 = 4$$

But y is not defined $\forall x$ ∴ what if $x^2 > 4$? $\Rightarrow$ we can't put this in standard form.

Objects we won't be studying

There are some ODEs we can't put into standard form $x = f(x,t)$.

$$\text{e.g.} \quad x^2 + \dot{x}^2 = 4.$$

Whilst this satisfies our def^n of an ODE, but if we solve for $\dot{x}$, we get

$$\dot{x} = \pm\sqrt{4-x^2}$$

$\Rightarrow$ $\dot{x}$ may take 2 values or no value. Such eq^{n}s do arise in SCIENCE!! but are hard to solve.

Also macaroni.

Reducing higher order ODEs to standard form

Consider a 2nd order ODE such as

$$\ddot{x} + 2x\dot{x} + x^3 \sin t = 0.$$

If we write $y = \dot{x}$, we get

$$\dot{y} + 2xy + x^3 \sin t = 0.$$

These eq^{n}s define the 2D 1st order system

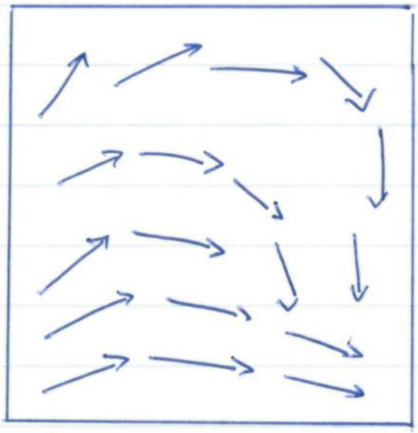
$$\dot{x} = y$$
$$\dot{y} = -2xy - x^3 \sin t$$

This first order system is exactly equivalent to the second order eq^n. All we have done is renamed $\dot{x}$ y. This process can be carried out for ODEs of any order.

Vector fields

There is a reason why we like ODEs of the form $\dot{x} = f(x,t)$. This is because they define vector fields. We can think of f as follows: at any point $x \in X$ and any particular

t, $f(x,t)$ defines a vector in $\mathbb{R}^n$. The size and direction of this vector tells us how the point x is going to 'evolve'.

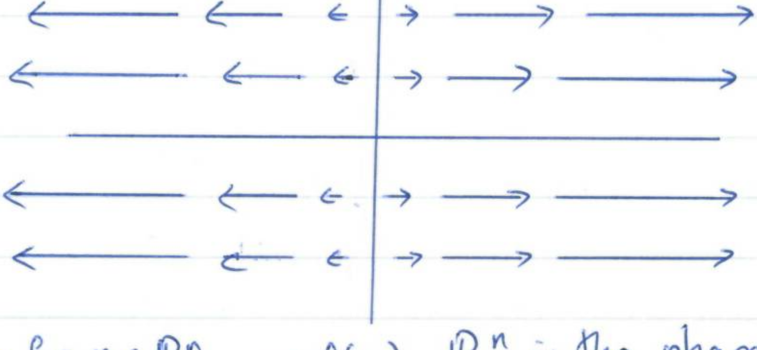


In the nonautonomous case, this vector field is constantly changing in time. In the autonomous case it remains the same throughout time. This makes autonomous ODEs much easier to study.

In 2D: $\left.\begin{array}{l}\dot{x} = x \\ y = 0\end{array}\right\}$ is an autonomous system defining a constant (in time) vector field $\binom{x}{0}$.

$\left.\begin{array}{l}\dot{x} = x \\ y = t\end{array}\right\}$ is a nonautonomous system defining a vector field $\binom{x}{t}$ which changes in time.

<u>Example</u>: Sketch the vector field given by $\left\{\begin{array}{l}\dot{x} = x \\ \dot{y} = 0\end{array}\right\}$ in the x-y plane



phase plane

Note for $x \in \mathbb{R}^n$, $x = f(x)$, $\mathbb{R}^n$ is the phase space and here it is $\mathbb{R}^2$ phase plane.

Solutions

A solution of an ODE is any function of the independent variable which satisfies the differential eq^{n}.

In general, $\phi(t)$ is a solution of the DE $\dot{x} = f(x,t)$

if $\frac{d\phi(t)}{dt} = f\big(\phi(t), t\big)$

To confirm that ϕ is a solution, we simply substitute in, and confirm that the LHS = RHS. (really!)

Example: The function $\phi(t) = \sin t$ is a solution to the ODE $\ddot{x} + x = 0$, as we can quickly check by direct substitution.
The function $\psi(t) = \cos t$ is also a solution, as is any linear combination of ϕ and ψ.

We can think also of solutions geometrically by thinking of the image of a sol^{n}. The function $\phi(t)$ which is a solution to an ODE tells us where the initial condition $\phi(0)$ moves to, forward or backward in time. Thus if $\phi(0) = x$, then $\phi(1)$ would tell us where x has 'moved to' after 1 second, etc.
Equally $\phi(-1)$ tells us where x 'was' at -1 seconds.
$\phi(t)$ may not exist for all time: x could shoot off to infinity very rapidly.

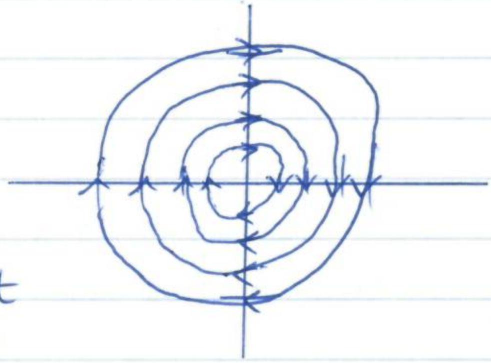
Assuming that ϕ is defined on some interval in $\mathbb{R}$, then the image of this interval under $\phi(t)$ is called a flow line or orbit or trajectory

Example: Phase plane for $\ddot{x} + x = 0$

Put $\ddot{x} + x = 0$ into standard form:

$$\dot{x} = y$$
$$\dot{y} = -x$$

Sketch the vector field:



We know that the general solution is $x = A\sin t + B\cos t$

$$y = A\cos t - B\sin t$$
$$\Rightarrow x^2 + y^2 = A^2 + B^2$$

$\Rightarrow$ orbits are circles centred at O.

Or, flow $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}y\\ -x\end{pmatrix}$ is $\perp^r$ to radius vector $\begin{pmatrix}x\\ y\end{pmatrix}$

Tangent to orbits

Tangent of the curve (orbit) is always $\perp^r$ to the radius vector $\Rightarrow$ circle centred at O.

Orbits

Let $\phi(t)$ be the solution satisfying $\phi(0) = x$. The forward orbit (or 'forward trajectory') through x is the set

$$\gamma^+(x) = \bigcup_{t \geq 0} \phi(t).$$

The backward orbit through x is $\gamma^-(x) = \bigcup_{t \leq 0} \phi(t)$.

The orbit through x is $\gamma^+(x) \cup \gamma^-(x) = \gamma(x)$

In each case, the domain of t is taken to be the time for which the sol^{n}s exists.

Existence and uniqueness

Given an ODE $\dot{x} = f(x)$ or $\dot{x} = f(x,t)$, we have to put some conditions on the f^n f near x to ensure there is exactly one solution $\phi(t)$ with $\phi(0) = x$. We will always assume that f is C^1, ensuring that exactly one sol^n passes through any initial condition. (However it is still possible that that the solution will exist only for a finite time.

In other words:

Theorem: Local existence and uniqueness

> Suppose $\dot{x} = f(x,t)$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuously differentiable. Then $\exists$ maximal $t_1 > 0$ and $t_2 > 0$ s.t. a sol^n $x(t)$ with $x(t_0) = x_0$ exists and is unique for all $t \in (t_0 - t_1, t_0 + t_2)$ where t_1 and t_2 could be ∞.

Proof not given.

Example: to show non-uniqueness of solutions if the C^1 property is relaxed, consider $x \in \mathbb{R}$ with $\dot{x} = f(x)$, $x(0) = 0$ where
$$f(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

Straightforward to verify that
$$x_T(t) = \begin{cases} 0 & \text{if } t < T \\ \frac{1}{4}(t-T)^2 & \text{if } t \geq T \end{cases}$$
is a sol^n $\forall\, T > 0$.

<u>Example</u> to show that the sol^{n} may only exist for finite time (if f is C^1) however smooth f may be:

$$\dot{x} = x^2.$$

which has sol^{n} with $x(0) = x_0$ given by $\frac{1}{x_0^{-1} - t} = x$

This f blows up in finite time if $x_0 > 0$ and in finite negative time if $x_0 < 0$.

However, the sol^{n}s that do exist are unique for the given initial conditions

<u>Flows</u>

It's often convenient to suppose that solutions exist $\forall t$. Suppose we know this about the autonomous ODE $\dot{x} = f(x)$. For each point $x \in X$ we have a solution $\phi_x(t)$ with $\phi_x(0) = x$. Put together all these solutions and we can think of them as a single function $\Phi(x,t) = \phi_x(t)$.

This f^{n} $\Phi(x,t)$ is called the <u>general sol^{n}</u> or <u>flow</u> generated by the DE. Clearly it satisfies the DE, ie.

$$\frac{\partial}{\partial t}\Phi(x,t) = f(\Phi(x,t))$$

For fixed $x = x_0$, $\Phi(x,t)$ is just the sol^{n} of the DE at time t with initial value x_0.

For fixed $t = t_0$, $\Phi(x,t_0)$ is a map which tells us where every point will evolve to after time t_0.

You can visualise a flow for an autonomous ODE as follows:

Imagine the space X covered with directed flow lines passing through every point and never intersecting. If you want to know how a point evolves, simply follow the flow line through that point. This picture tells you everything about where the points go in forward and backward time. It tells you nothing about how <u>fast</u> they move.

Following a flow line for t seconds and then another s seconds is the same as following it forward for t+s secs (Jesus...), so:

$$\Phi(x,t+s) = \Phi(\Phi(x,t),s)$$
$$= \Phi(\Phi(x,s),t).$$

There are two ways of using $\Phi(x,t)$:

- by choosing x and asking how it evolves in time
- by choosing a time t and asking where every point has got to at time t.

When thinking about $\Phi(x,t)$ in the second way, it is common to write $\Phi_t(x)$ instead of $\Phi(x,t)$ where each Φ_t is a map on X.

So a continuous-time dynamical system defines a <u>family of discrete-time dynamical systems</u>, one for each value of t. This is useful because we can use ideas already developed for discrete-time dynamical systems.

In this notation, $\Phi_{t+s}(x) = \Phi_t(\Phi_s(x))$
$= \Phi_s(\Phi_t(x))$

and $\Phi_0(x) = x$.

e.g. setting $f = \Phi_t$ defines a discrete-time dynamical system

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad x \longmapsto f(x) = \Phi_t(x)$$
$$\text{or } f : X \longrightarrow \mathbb{R}^n$$

and $f^n(x) = \Phi_{nt}(x)$

Parallel between discrete and continuous dynamical systems

<u>Def^n</u>: In general, a <u>dynamical system</u> is a manifold $\mathcal{M}$ (e.g. $\mathbb{R}^n$) called the phase space, endowed with a family of smooth evolution functions Φ_t, that for any element $t \in T$ the time, map of point of the phase space back into the phase space.

When T is the reals, this is a continuous dynamical system.
- - - - - - - integers, ... discrete - - - - - - .
(cf. $f, f^2, f^3 \ldots$)

Limit sets for a flow

A set M is <u>invariant</u> iff for all $x \in M$, $\Phi(x,t) \in M$ $\forall t$.
A set is <u>fwd/bkwd invariant</u> iff $\forall x \in M$, $\Phi(x,t) \in M$ $\forall t > 0$ or $\forall t < 0$ resp.
Similarly to the case for maps, the <u>ω-limit set</u> of x, $\omega(x)$, is defined as

$$\omega(x) = \{ y \in \mathbb{R}^n : \exists \text{ sequence } (t_n) \text{ with } t_n \to \infty \text{ and } \Phi_{t_n}(x) \to y \text{ as } n \to \infty \}$$

and the α-limit set of x, $\alpha(x)$ is:

$$\alpha(x) = \{ y \in \mathbb{R}^n : \exists (t_n) \text{ s.t. } t_n \to -\infty \text{ and } \Phi_{t_n}(x) \to y \text{ as } n \to \infty \}$$

$\omega(x)$ and $\alpha(x)$ are both <u>invariant</u> sets.

Moreover, by the Bolzano-Weierstraß Thm, if the trajectory of x in fwd/bkwd time is bounded, then they are always <u>non-empty</u>. They consist of points that the trajectory of x approaches in fwd/bkwd time respectively.

<u>Example:</u> $\dot{r} = r(1-r^2)$ $\quad$ $\mathbb{R}^2$ polar coordinates
$\dot{\theta} = 1$

$$r \neq 0, \quad \omega(r,\theta) = \{(r,\theta) : r=1\}$$
$$\alpha(r,\theta) = \begin{cases} \{(r,\theta) : r=0\} & 0<r<1 \\ \{(r,\theta) : r=1\} & r=1 \\ \text{undefined} & r>1 \end{cases}$$

<u>Proof that $\omega(x)$ is invariant</u>

$\omega(x) = \{y \in \mathbb{R}^n : \exists \text{ a sequence } (t_n) \text{ with } t_n \to \infty \text{ and } \Phi(x,t_n) \to y \text{ as } n\to\infty\}$

$y \in \omega(x) \Rightarrow \exists\, (t_n)$ s.t. $t_n \to \infty$ and $\Phi(x,t_n) \to y$ as $n \to \infty$.

Now fix t and consider (t_n') where $t_n' = t_n + t \ \forall n$
Clearly $(t_n') \to \infty$ as $n \to \infty$ (since t is fixed)

$$\Phi(x,t_n') = \Phi(\Phi(x,t_n),t) = \Phi_t(\Phi(x,t_n)).$$

Φ_t is continuous as a f^n of x and $\Phi(x,t_n) \to y$ as $n \to \infty$.

$$\Rightarrow \underbrace{\Phi_t(\Phi(x,t_n))}_{= \Phi(x,t_n')} \longrightarrow \Phi_t(y) \text{ as } n \to \infty$$

$$\Rightarrow \Phi_t(y) \in \omega(x)$$

But this is true for any $t \Rightarrow y \in \omega(x) \Rightarrow \Phi(y,t) \in \omega(x) \ \forall t$

$\Rightarrow \omega(x)$ is invariant $\square$.

(similarly for $\alpha(x)$).

Claim: If $\gamma^+(x)$ is bounded, $\omega(x)$ is nonempty

Proof: Suppose $x \in \mathbb{R}^n$, $\gamma^+(x)$ is bounded.
Consider (t_n) with $t_n = n \quad \forall n \geq 0$

$$\Phi_{t_n}(x) \in \gamma^+(x) \qquad \forall t_n$$

$\Rightarrow$ the sequence $(\Phi_{t_n}(x))$ is bounded in $\mathbb{R}^n$
$\Rightarrow$ by BWT, $\Phi_{t_n}(x)$ has a convergent subsequence, whose limit is thus in $\omega(x)$.
So $\omega(x)$ is nonempty. $\square$

Every limit set is the image of some special solution. Here are some examples:

Constant solutions

These are solutions of the form $\phi(t) = x_0$. Thus the initial point x_0 does not move at all. If $\phi(t) = x_0$ is a constant solution to an ODE, then x_0 is a fixed point of the associated flow, i.e. $\Phi(x_0, t) = x_0 \; \forall t$. The words equilibrium and steady state are often used to refer to such points. Constant solutions occur when $f(x) = 0$, i.e. the vector field vanishes at x.

Periodic solutions

A periodic solution is one which satisfies $\phi(t) = \phi(t+T) \; \forall t$ and for some value $T > 0$. This means that the flow line comes back to its starting point. Geometrically, it is a closed curve known as a periodic orbit.

A point x_0 is periodic of (minimal) period T
iff $\Phi(x_0, t+T) = \Phi(x_0, t) \quad \forall t \quad [\text{and } \Phi(x_0, t+S) \neq \Phi(x_0, t) \; \forall t \; \forall 0 < S < T]$

A constant solution is a periodic solution, and a nonconstant periodic solution will be called 'nontrivial'. If the system is autonomous, then we can only have a nontrivial periodic solution if the phase space has dimension 2 or greater (figure out why...)

Quasiperiodic solutions

A periodic solution has a single period, and returns to its starting value after this period. A quasiperiodic solution is, roughly speaking, a solution with two or more periods which are not rationally related (e.g. $\sin t + \sin(\sqrt{2}t)$). Such a solution traces out a torus in phase space.

Chaotic solutions

We will briefly discuss chaos in ODEs later, but not in the same detail as for maps.

Note: Apart from constant solutions, there ~~are~~ is no general way of identifying the kinds of special solution by looking at the vector field.

Note: (for autonomous systems) You can think of orbits as embeddings of lines in $\mathbb{R}^n$. By uniqueness, the orbit cannot cross itself.

In (a) a 1D autonomous system, an orbit can tend monotonically to a fixed point, or diverge monotonically to $\pm\infty$.

(b) a 2D autonomous system, an orbit can end up at a fixed point, diverge or tend to a closed curve (periodic orbit). Proved in week 9.

1D autonomous systems

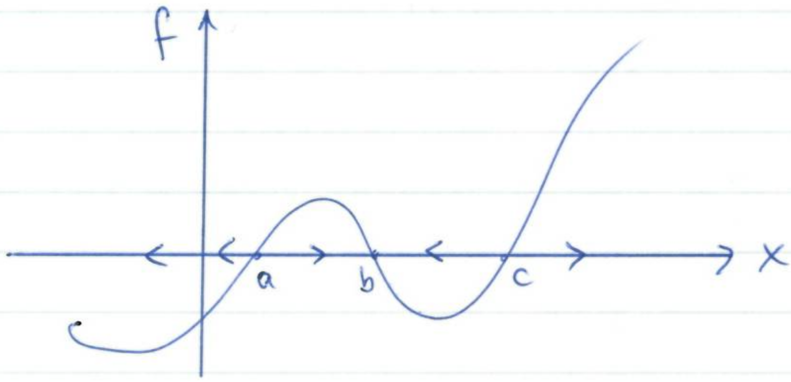
One dimensional autonomous differential eqⁿs are very simple to understand. Consider a 1D ODE, $\dot{x} = f(x)$, where $x \in \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$. We can, in principle, integrate the ODE as follows:

$$\int_{x_0}^{x(t)} \frac{1}{f(x)}\,dx = \int_0^t dt$$

although the integral on the LHS may be hard to evaluate.

Whether or not we can find solutions, if we can plot $f(x)$, then we can characterise the behaviour of the ODE completely. Whenever $f(x) > 0$, this means $\dot{x} > 0$, i.e. the vector field points to the right. When $f(x) < 0$, the vector field points to the left.

If $f(x) = 0$, we have an equilibrium.



Thus,
In this example, there are 3 equilibria at a, b & c.
For $x < a$, $f(x) < 0$ and any initial condition points left.
For $a < x < b$, $f(x) > 0$ and any initial condition moves right (toward b).
For $b < x < c$, $f(x) < 0$ and any initial condition moves left (toward b).
For $x > c$, $f(x) > 0$ and any initial condition moves right.

Thus we can tell immediately that a and c are unstable, while b is stable

Example: Consider the nonlinear DE:

$$\dot{x} = \sin x .$$

This is used to show how pictures can be more useful than formulae:

We can solve it by separating variables and integrating:

$$\int dt = \int \frac{1}{\sin x}\, dx = \int \operatorname{cosec} x \, dx$$

$$\Rightarrow \quad t = -\ln|\operatorname{cosec}(x) + \cot(x)| + C$$

Suppose $x = x_0$ at $t=0$, then $C = \ln|\operatorname{cosec} x_0 + \cot x_0|$

$$\Rightarrow \quad t = \ln\left|\frac{\operatorname{cosec}(x_0) + \cot(x_0)}{\operatorname{cosec}(x) + \cot(x)}\right| .$$

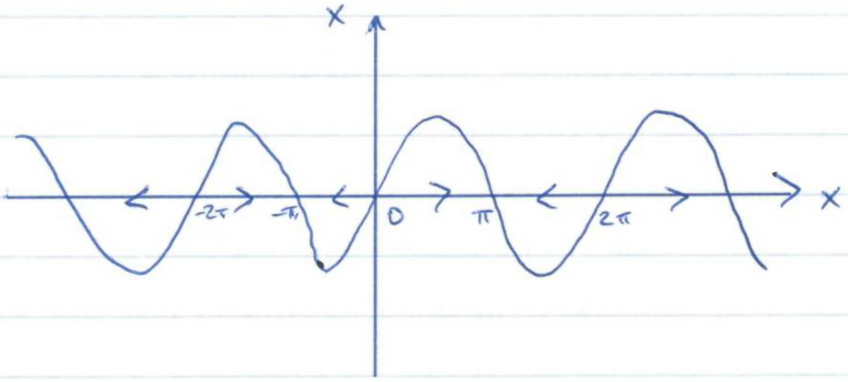
This result is correct but difficult to interpret. For example, can you answer the following questions?:

1. Suppose $x_0 = \frac{\pi}{4}$, describe the qualitative features of the solution $x(t)$ $\forall\, t > 0$. In particular, what happens as $t \to \infty$?

2. For an arbitrary initial condition x_0, what is the behaviour of $x(t)$ as $t \to \infty$?

DIFFICULT!!

So let's use graphical analysis (GA for Windows)

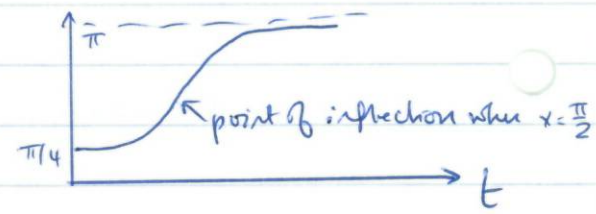


It is easy to see the sign of $\dot{x}$ and hence the dir$^{\underline{n}}$ of evolution for each x.

The fixed points satisfy $\sin(x) = 0 \Rightarrow x = n\pi, \ n \in \mathbb{Z}$ and it's clear from the figure that $x = 2n\pi, \ n \in \mathbb{Z}$ are unstable and $x = (2n+1)\pi, n \in \mathbb{Z}$ are stable.

<u>Answers</u> : 1. Starting at $x_0 = \frac{\pi}{4}$, the orbit moves to the right faster and faster, until it crosses $x = \frac{\pi}{2}$, then it starts slowing down and eventually approaches $x = \pi$ from the left.

e.g. sketch $x(t)$ with $x_0 = \pi/4$



2. For arbitrary x_0, if $x_0 = 2n\pi$ it stays there, otherwise it approaches the nearest stable fixed point ($\in \{(2n+1)\pi : n \in \mathbb{Z}\}$) monotonically as $t \to \infty$.

Stability

We have seen by example that an equilibrium a in our 1D autonomous system is stable if nearby initial conditions move towards the equilibrium. Locally we have two situations:

if $f'(a) < 0$, then the eq^m will be stable

if $f'(a) > 0$, it is unstable

Later, when we study bifurcation theory, we will see what can happen in a family of differential eq^{n}s when ~~the~~ $f'(a) = 0$, the non-hyperbolic case.

Note: in continuous dynamical systems, fixed points occur at
$f(x) = 0 \Rightarrow \Phi_t(x) = x \quad \forall t$
unlike discrete dynamical systems when fixed points occur at
$f(x) = x$

[DO NOT CONFUSE !!]

Stable fixed points: $f(x_0) = 0$ with $f'(x_0) < 0$.

Close to the fixed point: $x = x_0 + \delta \qquad (\delta \text{ small})$

$$\begin{aligned} \dot{\delta} = \dot{x} = f(x) &= f(x_0 + \delta) \\ &= \cancelto{0}{f(x_0)} + \delta f'(x_0) + O(\delta^2) \\ \dot{\delta} &\approx f'(x_0)\delta \end{aligned}$$

Locally, the perturbation from the fixed point approximately satisfies a linear homogeneous ODE.

In general, a 1D linear homogeneous ODE is $\dot{x} = ax$, with $x \in \mathbb{R}$ and $a \sim$ const.

This has solution $x = x(0)e^{at}$

$\Rightarrow$ $x \to 0$ as $t \to \infty$ if $a < 0$

$x \to \infty$ as $t \to \infty$ if $a > 0$

For the nonlinear case, $\delta \to 0$ as $t \to \infty$ if $f'(x_0) < 0$.

$\Rightarrow$ x_0 is stable if $f'(x_0) < 0$

n-dimensional ODEs

Written in standard form, the most general linear ODE takes the form
$$\dot{x} = \underbrace{A(t)}_{\text{time dependent matrix}} x + \underbrace{B(t)}_{\text{time dependent vector}}$$

If $B(t) \equiv 0$, this ODE is called <u>homogeneous</u>, else <u>inhomogeneous</u>.

Linear homogeneous ODEs

These take the form $\underline{\dot{x} = A(t)x}$. We will mostly be interested in the case where A is a constant matrix. Linear homogeneous ODEs have the following special property:

> if $\phi(t)$ and $\psi(t)$ are two solutions of $\dot{x} = A(t)x$, then so is $\alpha\phi(t) + \beta\psi(t)$ for any constants α and β (exercise to prove).

Taylor expansions

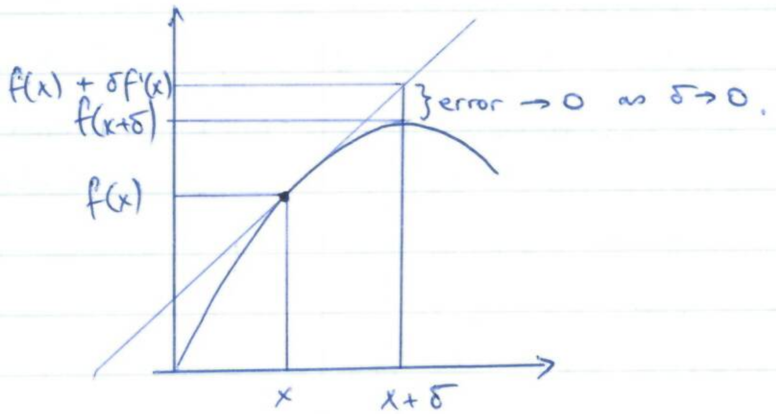
From basic calculus, we know that, as long as a f^n^ $f: R \to R$ is sufficiently differentiable, it can be expanded as a Taylor series:
$$f(x+\delta) = f(x) + \delta f'(x) + \frac{\delta^2}{2!} f''(x) + \cdots,$$

One way of looking at this eq^n is that we get successive approximations to the values of f at the point $x+\delta$.

The first order approximation is

$$f(x+\delta) \simeq f(x) + \delta f'(x)$$

which is a linear extrapolation and has an error which tends to zero like $|\delta|^2$ as $\delta \to 0$.



Taylor expansions can also be done for higher dimensional real functions on $\mathbb{R}^n$, in which case the derivative or Jacobian is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$.

Why linear systems are important for studying nonlinear systems

The Taylor expansion of a differentiable f^n $f: \mathbb{R}^n \to \mathbb{R}^n$ takes the form near a point y

$$f(y+\delta) = f(y) + Df(y)\,\delta + O(|\delta|^2)$$

where $Df(y)$ is the Jacobian evaluated at y.

e.g. $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

$$Df(x,y) = \begin{pmatrix} \left.\frac{\partial f_1}{\partial x}\right|_{(x,y)} & \left.\frac{\partial f_1}{\partial y}\right|_{(x,y)} \\ \left.\frac{\partial f_2}{\partial x}\right|_{(x,y)} & \left.\frac{\partial f_2}{\partial y}\right|_{(x,y)} \end{pmatrix}$$

So if $\delta = \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix}$

then $f_1(x+\varepsilon, y+\delta) = f_1(x,y) + \varepsilon \frac{\partial f_1}{\partial x}\Big|_{(x,y)} + \delta \frac{\partial f_1}{\partial y}\Big|_{(x,y)} + O(|\delta|^2)$

So if $x = y + \delta$ and δ is small, then $f(x) \approx f(y) + Df(y)\,\delta$.

Consider an autonomous DE $\dot{x} = f(x)$. Suppose we know a particular solution $\phi(t)$ with $\phi(0) = y$. What can we say about evolution of conditions near y? To answer this question, we examine the solution $\phi(t) + \delta(t)$ which takes the value $y + \delta(0)$ at a time $t = 0$.

Substituting $\phi(t) + \delta(t)$ into the ODE gives:

$$\frac{d}{dt}\left[\phi(t) + \delta(t)\right] = f\left[\phi(t) + \delta(t)\right]$$

Taylor expansion gives:

$$\dot{\phi}(t) + \dot{\delta}(t) = f[\phi(t)] + Df[\phi(t)]\,\delta(t) + O\left[|\delta(t)|^2\right]$$

Since $\phi(t)$ is itself a solution, $\dot{\phi}(t) = f[\phi(t)]$

so $\dot{\delta}(t) = Df[\phi(t)]\,\delta(t) + O(|\delta|^2)$.

The argument tells us so far how $\delta(t)$, the small perturbation, evolves. The next argument is approximate: as long as $|\delta|$ is sufficiently small, then its evolution satisfies (approximately) the <u>linear</u> homogeneous DE

$$\dot{\delta} = Df[\phi(t)]\,\delta$$

Of course, for any $\delta(0)$, however small, it is possible that eventually $\delta(t)$ will be so large that the linear approximation is no good. Still, at least for a short time, the linear approximation will give us a good estimate of what happens to small perturbations to $\phi(t)$.

So, linear differential eq^n's can tell us how small perturbations to solutions evolve.

If the solution $\phi(t)$ is periodic, then $Df[\phi(t)]$ is a time periodic matrix. If $\phi(t) = x_0$ is a constant solution (i.e. x_0 is an equilibrium) then $Df[\phi(t)] = Df(x_0)$ is a constant matrix and the ODE $\dot{\delta} = Df(x_0)\,\delta$ which tells us how i.c.'s near x_0 evolve is autonomous and linear.

Example Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f_1(x,y) = x^2 - 2y$
$f_2(x,y) = 3x^3 y$

$(0,0)$ is an eq^m of the associated ODE

The Jacobian Df is just the matrix of partial derivatives,

$$Df = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix}$$

$$= \begin{pmatrix} 2x & -2 \\ 9x^2 y & 3x^3 \end{pmatrix}$$

and $Df(0,0) = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$

So if $\delta(t) = \begin{pmatrix} \varepsilon(t) \\ \delta(t) \end{pmatrix}$, then

$$\frac{d}{dt}\begin{pmatrix}\varepsilon(t)\\ \delta(t)\end{pmatrix} \approx Df(0,0)\begin{pmatrix}\varepsilon(t)\\ \delta(t)\end{pmatrix}$$

i.e. $\dot{\varepsilon} \approx -2\delta$

$\dot{\delta} \approx 0$

Solving linear autonomous systems

Autonomous ODEs of the form $\dot{x} = Ax$ on $\mathbb{R}^n$ can be explicitly solved. If A were a scalar, then we know that there would be a solution of the form $x(0)\,e^{At}$. In this case A is an $n \times n$ matrix. As for a real number, the exponential of a square matrix is defined as:

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

By analogy with the scalar case, we define $T^0 = Id$. e^T is an $n \times n$ square matrix because it's the sum of sq. $n \times n$ matrices.

[Technical note: we have not proved that the series converges (indeed, we haven't said what it means for matrix sums to cvg.) This would involve defining a norm $\|\cdot\|$ on the space of matrices, and then showing $\|T^k\| \leq \|T\|^k$. After this this proof becomes identical to the real n°s.]

Having defined e^T, we can check by direct substitution that the vector $e^{tA}x_0$ solves the linear DE $\dot{x} = Ax$ with initial condition x_0.

Substituting in the series representation gives:

$$\frac{d}{dt}\left(e^{tA}\right) = \sum_{k=0}^{\infty} \frac{d}{dt}\left(\frac{t^k A^k}{k!}\right)$$

$$= \sum_{k=1}^{\infty} \frac{t^{k-1} A^k}{(k-1)!}$$

$$= A e^{kA}$$

We have made the assumption that we can differentiate the series term-by-term. Since the series which defines the exponential of a matrix is absolutely cvgt, this is OK. Thus the f$^{\underline{n}}$ $e^{tA}x_0$ gives the sol$^{\underline{n}}$ to the autonomous linear DE $\dot{x} = Ax$ which takes i.c. x_0.

We have just shown that:

<u>Thm</u>: The general sol$^{\underline{n}}$ to $\dot{x} = Ax$ is
$$\Phi(x,t) = e^{tA}x.$$

<u>Some results on the exponentials of matrices</u>

- In general, matrices don't commute.
  So preserve order, e.g.
  $$(S+T)^2 = S^2 + ST + TS + T^2$$
  $$(S+T)^3 = S^3 + S^2T + STS + ST^2 + TS^2 + TST + T^2S + T^3$$
  But if they <u>do</u>, $(S+T)^n = \sum_{r=0}^{n} \binom{n}{r} S^r T^{n-r}$

$\Rightarrow$

$$\Rightarrow e^{S+T} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} S^r T^{n-r}$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{S^r}{r!} \frac{T^{n-r}}{(n-r)!}$$
$$= \sum_{n=0}^{\infty} \sum_{r+k=n} \frac{S^r}{r!} \frac{T^k}{k!}$$
$$= \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{S^r}{r!} \frac{T^k}{k!}$$
$$= \left(\sum_{r=0}^{\infty} \frac{S^r}{r!}\right)\left(\sum_{k=0}^{\infty} \frac{T^k}{k!}\right) = e^S e^T.$$

1) If S and T commute, $e^{S+T} = e^S e^T$

2) $e^T e^{-T} = I$ (T and $-T$ always commute)
$\Rightarrow (e^T)^{-1} = e^{-T}$

3) If P and T are matrices, s.t. P invertible and $S = PTP^{-1}$, then $e^S = Pe^TP^{-1}$

Proof: $S = PTP^{-1} \Rightarrow S^2 = PTP^{-1}PTP^{-1} = PT^2P^{-1}$
And by induction, $S^n = PT^nP^{-1}$

$$\text{So } e^S = \sum_{n=0}^{\infty} \frac{1}{n!} S^n = \sum_{n=0}^{\infty} \frac{1}{n!} PT^nP^{-1}$$
$$= P\left[\sum_{n=0}^{\infty} \frac{T^n}{n!}\right] P^{-1}$$
$$= Pe^TP^{-1} \qquad \square.$$

Exponentials of special matrices

Exponentials of some special matrices can be quickly calculated by calculating powers by induction.

1) <u>A diagonal / block-diagonal matrix.</u>

If
$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & a_n \end{pmatrix}$$

where the a_i are scalars or square matrices, then by induction,
$$A_k = \begin{pmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & a_n^k \end{pmatrix}$$

and by the defn. of the exponential of a scalar / square matrix
$$e^A = \begin{pmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & e^{a_n} \end{pmatrix}$$

So if we have a diagonal matrix A and the ODE
$$\dot{x} = Ax,$$
we can quickly solve it:
$$x = e^{At} x_0$$
$$= \begin{pmatrix} e^{a_1 t} & 0 & \cdots & 0 \\ 0 & e^{a_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_n t} \end{pmatrix} x_0$$

2) <u>An important 2×2 matrix</u>

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Let $\lambda = a+ib$, then $A = \begin{pmatrix} Re(\lambda) & -Im(\lambda) \\ Im(\lambda) & Re(\lambda) \end{pmatrix}$

Claim $A^k = \begin{pmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{pmatrix}$

<u>Proof</u>: Clearly true for $k=1$.
Suppose true for 1 to $n-1$.

$$A^n = \begin{pmatrix} Re\,\lambda & -Im\,\lambda \\ Im\,\lambda & Re\,\lambda \end{pmatrix} \begin{pmatrix} Re\,\lambda^{n-1} & -Im\,\lambda^{n-1} \\ Im\,\lambda^{n-1} & Re\,\lambda^{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} Re(\lambda)Re(\lambda^{n-1}) - Im(\lambda)Im(\lambda^{n-1}) & -Re(\lambda)Im(\lambda^{n-1}) - Im(\lambda)Re\lambda^{n-1} \\ Re(\lambda)Im(\lambda^{n-1}) + Im(\lambda)Re(\lambda^{n-1}) & Re(\lambda)Re(\lambda^{n-1}) - Im(\lambda)Im(\lambda^{n-1}) \end{pmatrix}$$

Now, $\lambda^n = \lambda\lambda^{n-1} = [Re\,\lambda + i\,Im\,\lambda][Re\,\lambda^{n-1} + i\,Im\,\lambda^{n-1}]$
$= [Re\,\lambda\, Re\,\lambda^{n-1} - Im\,\lambda\, Im\,\lambda^{n-1}$
$+ i\,Im\,\lambda\, Re\,\lambda^{n-1} + i\,Re\,\lambda\, Im\,\lambda^{n-1}$

$$\Rightarrow A^n = \begin{pmatrix} Re(\lambda^n) & -Im(\lambda^n) \\ Im(\lambda^n & Re(\lambda^n) \end{pmatrix}$$

By induction $\square$.

So $$e^A = \sum_{k=0}^{\infty} \begin{pmatrix} Re\left(\frac{\lambda^k}{k!}\right) & -Im\left(\frac{\lambda^k}{k!}\right) \\ Im\left(\frac{\lambda^k}{k!}\right) & Re\left(\frac{\lambda^k}{k!}\right) \end{pmatrix}$$

$$= \begin{pmatrix} Re(e^\lambda) & -Im(e^\lambda) \\ Im(e^\lambda) & Re(e^\lambda) \end{pmatrix}$$

$e^\lambda = e^{a+ib} = e^a(\cos b + i \sin b)$

$$\Rightarrow \quad e^A = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$$

Coordinate transformations

Consider the DE $\dot{x} = f(x)$. We can think of any differentiable invertible transformation of the form $y = g(x)$ as a coordinate transformation.

In the new coordinate system, the DE is

$$\dot{y} = Dg(x)\,\dot{x}$$

$$= Dg[g^{-1}(y)]\, f[g^{-1}(y)].$$

The RHS looks ugly but it is possible that by choosing the fn $g(x)$ sensibly, we may get a system which takes a very simple form. Importantly, all qualitative behaviors in the two systems, <u>including stability of orbits</u>, is the same.

<u>Example</u> Consider the following system in $\mathbb{R}^2$:

$$\dot{x} = y + x(1-x^2-y^2)$$
$$\dot{y} = -x + y(1-x^2-y^2).$$

Consider the transformation into polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad x^2+y^2 = r^2$$

Differentiating the last term gives

$$r\dot{r} = x\dot{x} + y\dot{y} = \begin{cases} xy + x^2(1-x^2-y^2) \\ -xy + y^2(1-x^2-y^2) \end{cases}$$

$$= r^2(1-r^2)$$

So $\dot{r} = r(1-r^2)$

Differentiating e.g. $x = r\cos\theta$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$r\sin\theta + r\cos\theta\,(1-r^2) = r(1-r^2)\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\underline{\dot{\theta} = -1}$$

This system is very simple because $\dot{r}$ depends only on r, $\dot{\theta}$ only on θ. It is easy to completely characterise the dynamics of the system.

An important class of coordinate transformations are linear transformations of linear systems

Linear transformations of linear systems

Consider the linear autonomous system $\dot{\underline{x}} = A\underline{x}$, where $x \in \mathbb{R}^n$ and A is an $n \times n$ matrix. Now let's carry out a linear coordinate transformation on this system.

We call our new coordinates y and let $y = Px$, where P is an invertible matrix. Then the DE for y is

$$\dot{y} = P\dot{x} = PAx = PAP^{-1}y$$

We see that it is again a linear autonomous ODE of the form $\dot{y} = By$, where $B = PAP^{-1}$.

However, B may be simpler to exponentiate than A !! 
so the ODE for y may be easy to solve.

Stability of the zero solution for linear autonomous ODEs

Trivially, the system $\dot{\underline{x}} = A\underline{x}$ always has eqm at $x = 0$. We state the main result and then illustrate it.

Thm: The zero solution of $\dot{\underline{x}} = A\underline{x}$ is stable if all the eigenvalues of A have negative real part.

Possibility 1: Diagonalisable matrices

Starting with $\dot{x} = Ax$, imagine we can apply a coordinate transformation $y = Px$, which diagonalises the system. We know how to exponentiate a diagonal matrix

$$\text{e.g. if } PAP^{-1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & a_n \end{pmatrix}$$

then the general solⁿ is

$$\begin{aligned} y_1(t) &= y_1(0)\, e^{a_1 t} \\ y_2(t) &= y_2(0)\, e^{a_2 t} \\ &\vdots \\ y_n(t) &= y_n(0)\, e^{a_n t} \end{aligned}$$

all these values will tend to zero iff all $a_i < 0$. But the a_i are simply the e'vals of PAP^{-1} and therefore also of A. $\square$

Possibility 2: Repeated eigenvalues

Unfortunately not every matrix can be diagonalized

 awwww  NOT ALL DREAMS CAN COME TRUE

Assume, after a coordinate transform, that the system takes the form $\dot{y} = By$, where $B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

Clearly B has the repeated eigenvalue a. If b is nonzero, there is no invertible matrix P s.t. PBP^{-1} is diagonal.
(check this for yourself – this is because B does not have a basis of e'vectors).

Nevertheless, for this matrix we can check that

$$e^B = e^a \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

The general solution therefore is

$$y(t) = e^{Bt} y(0) = e^{at} \begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix} y(0)$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_1(0) e^{at} + y_2(0)\, bt\, e^{at} \\ y_2(0) e^{at} \end{pmatrix}.$$

$\Rightarrow$ the zero solution is stable provided $a < 0$.
$\Rightarrow$ in this situation too, the theorem is true.
In a very similar way, one can calculate the general sol^n to the eq^n $\dot{y} = By$, where

$$B = \begin{pmatrix} a & b & 0 & \cdots & 0 \\ 0 & a & b & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & a \end{pmatrix}$$

This time the eigenvalue a has multiplicity > 2
Again, as long as a is negative, the zero sol^n is stable.

Possibility 3: Another type of matrix which cannot be diagonalised (over $\mathbb{R}^n$) is a matrix with complex eigenvalues.

Assume, after a change of coordinates, we have a system of the form $\dot{y} = By$, where
$$B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

$$e^B = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix},$$

$$\text{so} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{Bt} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$$

$$= e^{at} \begin{pmatrix} y_1(0)\cos bt - y_2(0)\sin bt \\ y_1(0)\sin bt + y_2(0)\cos bt \end{pmatrix}$$

Clearly if $a > 0$, all solutions tend to infinity as $t \to \infty$; and if $a < 0$, all solutions tend to zero as $t \to \infty$ $\Rightarrow$ the theorem holds
(a is real eigenvalues of B) $\square$

<u>Jordan Normal Form</u> - an important result from matrix theory

Given any square matrix A, there is a similarity transformation P which puts it into <u>block diagonal form</u>

$$PAP^{-1} = \begin{pmatrix} A_1 & 0 & \cdot & 0 \\ 0 & A_2 & \cdot & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & A_n \end{pmatrix}$$

where the A_i are square matrices, which are either

- a single real number
- a 2×2 matrix with a pair of complex conjugate eigenvalues
- an $m \times m$ block of the form

$$\begin{pmatrix} a & 1 & 0 & \cdots & 0 \\ 0 & a & 1 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & a \end{pmatrix}$$

  corresponding to a repeated eigenvalue a.
- a similar block corresponding to a repeated complex eigenvalue.

So if we know how to exponentiate these four kinds of matrix, then we can exponentiate any matrix, by first transforming into Jordan Normal Form.

Stability of the zero solution for linear autonomous ODEs

Trivially, the system $\dot{x} = Ax$ always has an equilibrium at the origin.

Theorem: The origin is asymptotically stable if all the eigenvalues of A have negative real part.

Sketch of proof: First perform a similarity transformation to bring A into Jordan Normal Form (This does not change the eigenvalues of A).

Then treat each block A_i separately and exponentiate to calculate the general solution for that subsystem. Then show that in each case solutions converge to zero if the eigenvalues of A_i are negative.

Return to local stability of equilibria in nonlinear systems

We have seen how Taylor expanding the RHS of a nonlinear ODE near an equilibrium can give us a linear ODE. We now complete the argument by stating when nonlinear ODEs display qualitatively the same dynamics as linear ODEs in a region.

Def^n: Hyperbolic and nonhyperbolic equilibria

An equilibrium $\bar{x}$ of an ODE $\dot{x} = f(x)$ is hyperbolic if none of the eigenvalues of $Df(\bar{x})$ have real part equal to zero. (i.e. do not lie in the imaginary axis). An equilibrium $\bar{x}$ is nonhyperbolic if some of the eigenvalues of $Df(\bar{x})$ have zero real part.

Hartman-Grobman theorem (local behaviour near equilibria).

Assume we have an autonomous ODE $\dot{x} = f(x)$ with a hyperbolic equilibrium at $\bar{x}$, ie. $Df(\bar{x})$ has no eigenvalues on the imaginary axis. Then the flow generated by the linear ODE $\dot{\delta} = Df(\bar{x})\delta$ is conjugate to the flow generated by $f(x) = \dot{x}$ near $\bar{x}$.

Proof: not given.

Local stability of equilibria in autonomous systems

Consider any autonomous ODE $\dot{x} = f(x)$ and assume that it has an equilibrium at $\bar{x}$, i.e. $f(\bar{x}) = 0$.

Then $\boxed{\bar{x} \text{ is asymptotically stable if all eigenvalues of } Df(\bar{x}) \text{ have negative real part}}$

Sketch of proof: We have seen that in the linear case, eigenvalues with negative real parts implies asymptotic stability. The H-G thm now states that the nonlinear ODE has essentially the same orbit structure (that's what it means to be conjugate) as the linear ODE near the equilibrium, since it is hyperbolic $\square$

$\boxed{\text{Similarly, } \bar{x} \text{ is \underline{unstable} if any eigenvalue of } Df(\bar{x}) \text{ has positive real part.}}$

Summary: The general process which we have carried out is:

TAKE A NONLINEAR ODE	$\longrightarrow$	LINEARISE NEAR SOME SOLUTION TO GET LINEAR ODE	$\longrightarrow$	EXAMINE LINEAR ODE	$\longrightarrow$	USE THE RESULT TO SAY SOMETHING ABOUT STABILITY IN THE NONLINEAR ODE

We have mainly focussed on the special case where:

- The sol^n of the nonlinear ODE is an equilibrium
- So the linear ODE we get is autonomous and can be exactly solved.

Example: The matrix $A = \begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$

has eigenvalues -5 and 2.

We can check the corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

So that $P^{-1} = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$ and $P = \frac{1}{7}\begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}$

with $PAP^{-1} = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}$.

Clearly the system $\dot{x} = Ax$ in the new coord. system is uncoupled and easy to solve.
The origin is of saddle type (unstable).

Example: 'Recoordinatise'
$$\dot{x} = -x$$
$$\dot{y} = -y + (1-x)e^x$$

using $u = x$
$v = y - e^x$

$$\dot{u} = \dot{x} = -x = -u$$
$$\dot{v} = \dot{y} - e^x\dot{x} = -y + (1-x)e^x - e^x(-x)$$
$$= -y + e^x = -v$$

Clearly, the zero steady state in (u,v)-space is stable (this corresponds to $x=0$ $y=1$)

Surfaces in phase space

Suppose we have a smooth surface in phase space S and we choose some $x \in S$. We can ask:

(1) Is the vector field at x tangent to S?

(2) Is the vector field at x normal to S?

Finding the normal and tangent to a surface

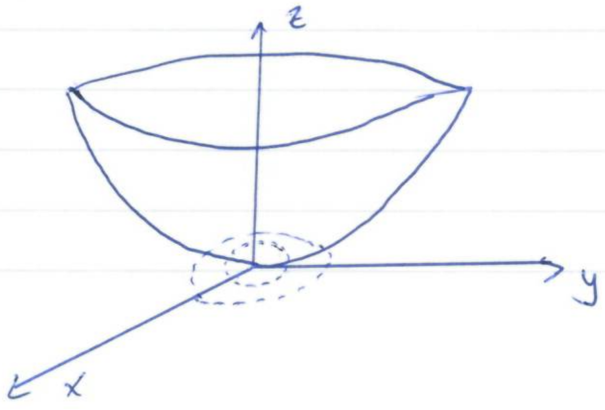
Consider a differentiable scalar f^n $V(x)$ on $\mathbb{R}^n$.
For each real number c we get a <u>level set</u> of V defined by $V(x) = c$ (think of the contours on a map/weather chart).
The normal to one of these level sets at x is given by

$$\text{grad}\, V = \nabla V = \begin{pmatrix} \partial/\partial x_1 (V) \\ \partial/\partial x_2 (V) \\ \vdots \end{pmatrix}$$

[$\nabla V(x)$ is the direction of steepest increase of V at x]

So we can decide if a vector field $\underline{F}$ is tangent to a surface defined by $V(x) = c$ by looking at ∇V and $\underline{F}$.
If the dot product $\nabla V \cdot \underline{F} = 0$, then $\underline{F}$ is tgt to $S(V=c)$.
If ∇V is collinear with $\underline{F}$, then $\underline{F}$ is normal to S.

The picture shows a plot of the scalar f^n $V(x, y) = x^2 + y^2$.
Its level sets are a series of circles centred on O.
It has a single minimum at O.



Example The vector field $\underline{F}$ on $\mathbb{R}^2$ defined by $\dot{x}=y,\ \dot{y}=-x$ is tangent to <u>every</u> surface defined by
$V(x) = x^2+y^2=r \quad (r>0)$ at every point.

To see this, we check that
$$\nabla V = \begin{pmatrix} 2x \\ 2y \end{pmatrix}, \quad \text{s.t. } \nabla V \cdot \underline{F} = 0$$

Example The vector field on $\mathbb{R}^2$ defined by $\dot{x}=x,\ \dot{y}=y$ is normal to every surface $x^2+y^2=r$ at every p^t because ∇F is collinear with $\underline{F}$.

Liapunov functions and global stability of equilibria

We saw that an eq^m is <u>locally</u> asymptotically stable if the eigenvalues of the Jacobian at the fixed p^t all have negative real parts. If the equilibrium is non-hyperbolic, or we want more global information about stability, we need to use some other methods.

The next technique, finding the Liapunov f^n, is one such method.

Def^n: Liapunov f^n: Suppose we have the vector field $\dot{\underline{x}} = f(\underline{x})$ which has an eq^m at $\bar{x}$.

A C^1 f^n $V: U \to R$ defined in some neighbourhood U of $\bar{x}$ s.t. $V(\bar{x})=0$ and $V(\underline{x})>0$ if $\underline{x} \neq \bar{x}$ is called a Liapunov f^n.

We can think of a Liapunov f^n as a surface with hills and valleys sitting above the space, with the deepest valley at $\bar{\underline{x}}$.

<u>Theorem (Liapunov Stability)</u>

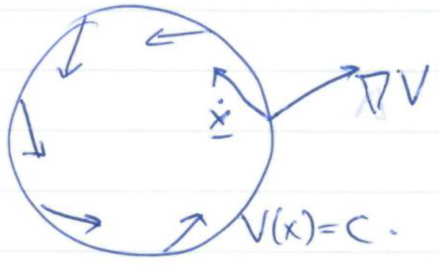
Suppose $\dot{\underline{x}} = f(\underline{x})$ has an eq^m at $\bar{\underline{x}}$.
Suppose that there is a Liapunov f^n on some neighbourhood U of $\bar{\underline{x}}$ satisfying $V(\bar{\underline{x}}) = 0$ and $V(\underline{x}) > 0 \ \forall \ \underline{x} \neq \bar{\underline{x}}$.

Then (i) if $\dot{V}(\underline{x}) \leq 0$ in $U \setminus \{\bar{\underline{x}}\}$, then $\bar{\underline{x}}$ is stable
(ii) if $\dot{V}(\underline{x}) < 0$ in $U \setminus \{\bar{\underline{x}}\}$, then $\bar{\underline{x}}$ is asymp. stable
(iii) if $\dot{V}(\underline{x}) > 0$ in $U \setminus \{\bar{\underline{x}}\}$, then $\bar{\underline{x}}$ is unstable.

$\dot{V}(\underline{x})$ means 'how V changes along a trajectory'
i.e. $\dot{V}(\underline{x}) = 0 \Rightarrow V$ is nonincreasing along any trajectory.
By the chain rule, $\dot{V}(\underline{x}) = \nabla V(\underline{x}) \cdot \dot{\underline{x}}$
But $\nabla V(\underline{x})$ defines the outward normal vector to the level set of V which goes through $\underline{x}$, and $\dot{\underline{x}}$ tells us about the vector field at the point $\underline{x}$.

The dot product of 2 vectors is negative or zero if the smallest angle between them is $\geq \pi/2$ $\quad (\because \underline{a} \cdot \underline{b} = ab\cos\theta)$

The condition $\nabla V \cdot \dot{\underline{x}} \leq 0$ means the vector field always points inwards on any level set of V.



This in turn means that any flow line can move inwards or remain on the level set, but not move outwards.

<u>Proof</u> omitted.

<u>Recap</u>: <u>stable</u>: A fixed pt $\bar{x}$ of the flow Φ_t is stable iff, given any neighbourhood $\tilde{V}$ of $\bar{x}$, $\exists$ a neighbourhood $\tilde{U}$ of $\bar{x}$, s.t. $\forall x \in \tilde{U}$, and $\forall t \geq 0$, $\Phi_t(x) \in \tilde{V}$.

<u>asymp. stable</u>: A fixed pt $\bar{x}$ of the flow Φ_t is asymp. stable iff, $\exists$ a neighbourhood $\tilde{U}$ of $\bar{x}$ s.t. $\forall x \in \tilde{U}$, $\lim_{t \to \infty} \Phi_t(x) = \bar{x}$.

In the theorem, note that the neighbourhood U is arbitrary and may not be small. Indeed, if the L. f^n is defined on the whole space, and we have $\dot{V} < 0$ everywhere, then every pt in space must converge to the eq^m. Thus L. f^{n}s can be used to prove global stability

<u>Example</u> Consider the vector field on $\mathbb{R}^2$,
$$\dot{x} = y$$
$$\dot{y} = -x + \epsilon x^2 y.$$

This has a nonhyperbolic eq^m at $(0,0)$.

$\left[J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \text{e'vals are } \lambda^2 + 1 = 0, \ \lambda = \pm i \right]$

Linearisation doesn't tell us about its stability.

Instead we use a L. f^n.

Let $V(x,y) = \dfrac{x^2+y^2}{2}$.

Then $V(0,0) = 0$ and $V(x,y) > 0$ in any neighbourhood of $(0,0)$. So

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \nabla V \cdot \dot{x}$$

$$= \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\nabla V \text{ as defn'd by } f^n} \cdot \underbrace{\begin{pmatrix} y \\ -x+\varepsilon x^2 y \end{pmatrix}}_{\dot{\underline{x}} \text{ as defined in eq}^n} = xy - yx + \varepsilon x^2 y^2 = \varepsilon x^2 y^2$$

$$\dot{V} = \varepsilon x^2 y^2$$

Thus for $\varepsilon < 0$, the fixed point is globally stable. Note that we have not proved asymptotic stability since $\dot{V} = 0$ along the lines $x=0$ and $y=0$, not just at the pt $(0,0)$.

In fact, the eqm. at the origin is asymptotically stable, but we would have to do a little more work to confirm this.

Invariant sets

Sometimes ODEs have invariant sets defined by some algebraic eq^n. Consider the set defined by the algebraic eq^n $g(x)=0$ and so suppose it is an invariant set for the ODE system, $\dot{x}=f(x)$.

This means that if we take an initial condition satisfying $g(x)=0$, then $\dot{g}(x)=0$, ie. g doesn't change along the trajectory.

By the chain rule, $\frac{d}{dt}\left[g(x(t))\right] = Dg(x)\,\dot{x} = Dg(x)\,f(x)$

$g(x)$ need not be a scalar eq^n., but could be any system of eq^{n}s and so $Dg(x)$ is in general a matrix.

If $Dg(x)\,f(x)$ is zero when evaluated at any point x on the surface $g(x)=0$, then the surface is indeed invariant.

Example: Consider the system

$$\dot{x} = y + x(1-x^2-y^2)$$
$$\dot{y} = -x + y(1-x^2-y^2)$$

We can confirm that the eq^n $x^2+y^2-1=0$ defines an invariant circle as follows:

Let $g(x,y) = x^2+y^2-1$ and differentiate

$$\dot{g} = \begin{pmatrix} 2x & 2y \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 2xy + 2x^2(1-x^2-y^2) - 2yx + 2y^2(1-x^2-y^2)$$
$$= 2(x^2+y^2)(1-x^2-y^2)$$

Evaluating $\dot{g}$ on $x^2+y^2=1$, we see that $\dot{g}=0$ on this circle. Thus the circle $x^2+y^2=1$ is invariant for this system.

Example. The Lorenz model is given by

$$\dot{x} = S(y-x)$$
$$\dot{y} = Rx - y - xz$$
$$\dot{z} = xy - Bz$$

Show that the z-axis is invariant.

$$g(x,y,z) = \underline{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left(\text{the } z\text{-axis is } \begin{matrix} x=0 \\ y=0 \end{matrix}\right)$$

$$Dg = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\dot{g} = Dg\,\dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} S(y-x) \\ Rx-y-xz \\ xy-Bz \end{pmatrix} = \begin{pmatrix} S(x-y) \\ Rx-y-xz \end{pmatrix}$$

and so on $g=0$ $\quad \dot{g} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow g=0$ is invariant.

Energy f^n's and conservative systems

Consider an ODE system $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$ and a scalar f^n $E(x)$ defined on $\mathbb{R}^n$.

The level sets of $E(x)$ define surfaces of dimension $n-1$. We know that the gradient $\nabla E(x)$ at any point is a vector normal to the level surface at that point.

Now suppose that the vector field is always tangent to the level sets of E at every point. In other words, suppose that it is always perpendicular to ∇E at every pt. This would imply that every flow line must lie on a level set of E.

Conservative systems in $\mathbb{R}^2$

Systems with an energy f^n in $\mathbb{R}^2$ are very easy to construct and understand. Let's suppose we have an energy f^n $E(x,y)$. Then ∇E is given by

$$\nabla E(x,y) = \begin{pmatrix} \partial E/\partial x \\ \partial E/\partial y \end{pmatrix}$$

For any constant k, the vector $\begin{pmatrix} k\,\partial E/\partial y \\ -k\,\partial E/\partial x \end{pmatrix}$ is normal to $\nabla E(x,y)$.

So the dynamical system

$$\dot{x} = k\,\partial E/\partial y$$
$$\dot{y} = -k\,\partial E/\partial x$$

will have its flow lines lying on the level sets of $E(x,y)$.

Example Let's choose a very simple energy f^n $E(x,y) = x^2+y^2$. The level sets of this f^n are circles centred at the origin.

We can check that $\nabla E(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$.

So the dynamical system
$$\dot{x} = ky$$
$$\dot{y} = -kx$$
has all its flow lines on surfaces of constant E.

Example: 3D conservative system.

Consider
$$\dot{x} = yz$$
$$\dot{y} = -z(x+1)$$
$$\dot{z} = y$$

Let $E(x,y,z) = x^2+y^2+z^2$
$$\nabla E = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\nabla E \cdot \underline{\dot{x}} = 2xyz - 2yz(x+1) + 2zy$$
$$= 0$$

$\Rightarrow$ $\dot{E} = 0$ along orbits. The orbits lie on spherical shells centred at O, $E = \text{const}$. (level surfaces of E).

Example: Suppose
$$\dot{x} = f(y)$$
$$\dot{y} = g(x)$$
Look for $E(x,y)$ s.t. $\dot{E} = \nabla E \cdot \underline{\dot{x}} = 0$.
$$\Rightarrow \nabla E = k\begin{pmatrix} g(x) \\ -f(y) \end{pmatrix}$$

$$\Rightarrow E = k\left[\int g(x)\,dx - \int f(y)\,dy + \text{const}\right]$$

So $\dot{x} = f(y)$

$\dot{y} = g(x)$

is conservative for any integrable f^{n}s f and g,

with $E(x,y) = \int g(x,\cancel{y})\,dx - \int f(\cancel{x}y)\,dy$.

= the conserved quantity.

Hamiltonian systems

Particular cases of conservative systems are <u>Hamiltonian systems</u>. These arise in various physical problems. They are systems on $\mathbb{R}^{2n}$ (i.e. phase spaces of even dimension), and are defined by the eq^{n}s

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}$$

Where $x, y \in \mathbb{R}^n$ and H is a scalar f^n called the <u>Hamiltonian</u> in $\mathbb{R}^2$

Note x and y are vectors so

$$\frac{\partial H}{\partial x} = \begin{pmatrix} \partial H/\partial x_1 \\ \vdots \\ \partial H/\partial x_n \end{pmatrix} \quad \text{and} \quad \frac{\partial H}{\partial y} = \begin{pmatrix} \partial H/\partial y_1 \\ \vdots \\ \partial H/\partial y_n \end{pmatrix}$$

Example

Consider the Hamiltonian f^n on $\mathbb{R}^4$ defined by

$$H(x,y) = (x_1^2 + x_2^2 + y_1^2 + y_2^2)/2.$$

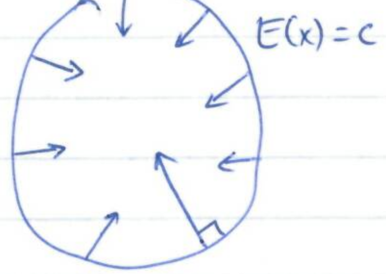
This is in fact the energy f^n for a spherical pendulum.

This gives rise to the dynamical system:

$$\left.\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{x}_2 &= y_2 \\ \dot{y}_1 &= -x_1 \\ \dot{y}_2 &= -x_2 \end{aligned}\right\}$$ This system consists of two independent simple harmonic oscillators

Gradient systems

These are a kind of 'opposite' to conservative systems. Again we assume that there is some scalar f^{n} $E(x)$ defined on the state space. Now, however, instead of requiring the vector field to be always tangent to level surfaces (so that $f(x) \cdot \nabla E = 0$), instead we require that the vector field is defined by ∇E, i.e. we can write the system $\dot{x} = -\nabla E(x)$.



The geometrical interpretation is that the vector field is always perpendicular to the level surfaces of E. This implies that E must strictly decrease along every trajectory except when the trajectory consists of an equilibrium. This strict condition severely restricts the dynamics of gradient systems.

Gradient vector fields arise in a variety of applications, e.g. in the study of electrical circuits.

It is easy to see that wherever E has a turning point (ie. $\nabla E = \underline{0}$), the vector field has an equilibrium.
If E has an isolated local minimum at $\bar{x}$, the E acts as a Liapunov f^{n} near $\bar{x}$, guaranteeing that $\bar{x}$ is asymptotically stable.

However, if E has an isolated local maximum at $\bar{x}$, then the eq^{m} is unstable.

Explicitly, we can calculate how E changes along trajectories:

$$\dot{E}(x) = \nabla E(x) \cdot \dot{x} = -|\nabla E(x)|^2$$

So $\dot{E}(x) \leq 0$ and $\dot{E}(x) = 0$ iff x is a turning pt of E. The fact that E must increase along any trajectory, unless the trajectory is an eq^{m}, rules out complicated limit sets, and forces the system to have very simple dynamics. In fact, the ω-limit set of any point (if it exists) can only consist of equilibria.

It is quite easy to see that a gradient system can have no more complicated limit sets. As we move around the limit set it is not possible for E to be strictly decreasing and yet for the orbit to come back arbitrarily close to itself.

Trajectories - more on the geometry of orbits

Summary of ideas on trajectories (orbits) and limit sets

Recall:

1. A trajectory is the image of a map from $\mathbb{R}$ to the state space (assuming sol^{n}s exist $\forall$ time)

$$\gamma(x) = \{\Phi_t(x)\}$$

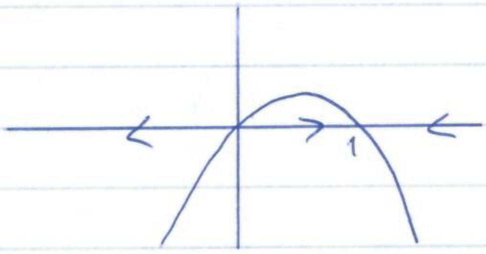
2. All points on a trajectory have the same trajectory

3. Trajectories of autonomous ODEs cannot intesect.

4. All points on a trajectory have the same limit sets.
(∴ it makes sense to talk about the limit sets of a trajectory)

<u>Example</u> $\dot{x} = x(1-x)$, $x \in \mathbb{R}$

- What is $\omega(x)$ for i) $x < 0$
  ii) $x = 0$
  iii) $x > 0$?



(i) undefined $\{-\infty\}$
(ii) $\{0\}$
(iii) $\{1\}$

- What is $\gamma(2)$? i.e. where did it come from, where did it go?
  $(1, \infty)$.

- What is $\gamma(-2)$? $(-\infty, 0)$.

Note $\forall x \in \gamma(2)$, $\omega(x) = \{1\}$
$\forall x \in \gamma(-2)$, $\Phi_t(x) \to -\infty$ as $t \to \infty$

- What is $\alpha(x)$ for i) $x < 0$ $\{0\}$
  ii) $x = 0$ $\{0\}$
  iii) $x > 0$ $0 < x < 1$ $\{0\}$
  $x > 1$ $\{\infty\}$ undefined
  $x = 1$ $\{1\}$

5. If a (forward) trajectory remains in a bounded region, then it must have an ω-limit set.

6. A point in the limit set of x need not be in the trajectory of x

Example: in $\dot{x} = x(1-x)$,

$$\omega(2) = \{1\} \text{ but } 1 \notin \gamma(2) = (1, \infty)$$

7. Limit sets consist of trajectories. If a point x is a member of a limit set, then so is its trajectory. Limit sets contain their own limit sets! (This means that given any pt x in a limit set Λ, the limit set of x is again in Λ.) This limit set may be a proper subset of Λ.)

Proof of first part of 7: Suppose $x \in \omega(y)$.

$\Rightarrow \exists\ (t_n)$ with $t_n \to \infty$ and $\Phi_{t_n}(y) \to x$.

Consider $z \in \gamma(x)$

$\Rightarrow z = \Phi_t(x)$ for some $t \in \mathbb{R}$.

Consider (s_n) with $s_n = t_n + t$, $\forall n$.

$\Phi_{s_n}(y) = \Phi_t(\Phi_{t_n}(y))$;

and $s_n \to \infty$ as $n \to \infty$.

By the continuity of Φ_t, $\Phi_{s_n}(y) \to \Phi_t(x) = z$, as $s_n \to \infty$. $\Rightarrow z \in \omega(y)$

$\Rightarrow x \in \omega(y) \Rightarrow \gamma(x) \subseteq \omega(y)$.

Similarly, $x \in \alpha(y) \Rightarrow \gamma(x) \subseteq \alpha(y)$. $\square$

Proof of second part of 7: Suppose $x \in \omega(y)$

$\Rightarrow \exists\ (t_n)$ with $t_n \to \infty$ and $\Phi_{t_n}(y) \to x$

Suppose $u \in \omega(x)$

$\Rightarrow \exists\ (v_n)$ with $v_n \to \infty$ and $\Phi_{v_n}(x) \to u$.

Let (w_n) be s.t. $w_n = t_n + v_n \quad \forall n$.

By continuity, $\Phi_{w_n}(y) \to u \Rightarrow u \in \omega(y)$

Similarly if $u \in \alpha(x)$, $x \in \alpha(y)$
then $u \in \alpha(y)$

$\Rightarrow$ Limit sets contain their own limit sets

A point which is in its own limit set

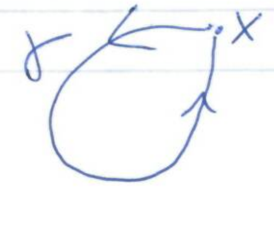
Note that if $x \in \gamma(x_0)$ and $x \in \omega(x_0)$, then $x \in \omega(x)$ since all points in $\gamma(x)$ have the same limit set.

8. Trajectories which are part of their own limit sets can be complicated: a trajectory that is part of its own limit set is a line which keeps returning arbitrarily near to itself at every point.

More precisely, suppose that a pt x is in its own limit set. This means that there is some sequence of times $t_j \to \infty$ s.t. $\Phi_{t_j}(x) \to x_0$.

Example: Homoclinic orbits

This is a special type of trajectory whose α- and ω-limit sets are the same. For example, the trajectory γ is homoclinic to the eq^{m} x.



This means that x is both the α-limit set and the ω-limit set of γ.

Note that γ is not a closed curve, but a closed curve minus one point. Note that the picture actually shows 2 trajectories (γ and x).

<u>Example: Heteroclinic orbits</u>

A trajectory which has an α-limit set A and an ω-limit set B is said to be a <u>heteroclinic orbit</u>. The trajectory γ is heteroclinic between the equilibria y and x. x is the ω-limit set and y is the α-limit set. γ is an open line segment.

Qu: how many trajectories are shown in the picture?

Ans: 3 (x, y, γ)

<u>Example of a heteroclinic/homoclinic orbit</u>

Recall the simple pendulum

$$ml\ddot{\theta} = -mg\sin\theta$$

If we rescale time, this becomes

$$\ddot{\theta} + \sin\theta = 0$$

Written as a dynamical system, we get

$$\dot{\theta} = v$$
$$\dot{v} = -\sin\theta$$

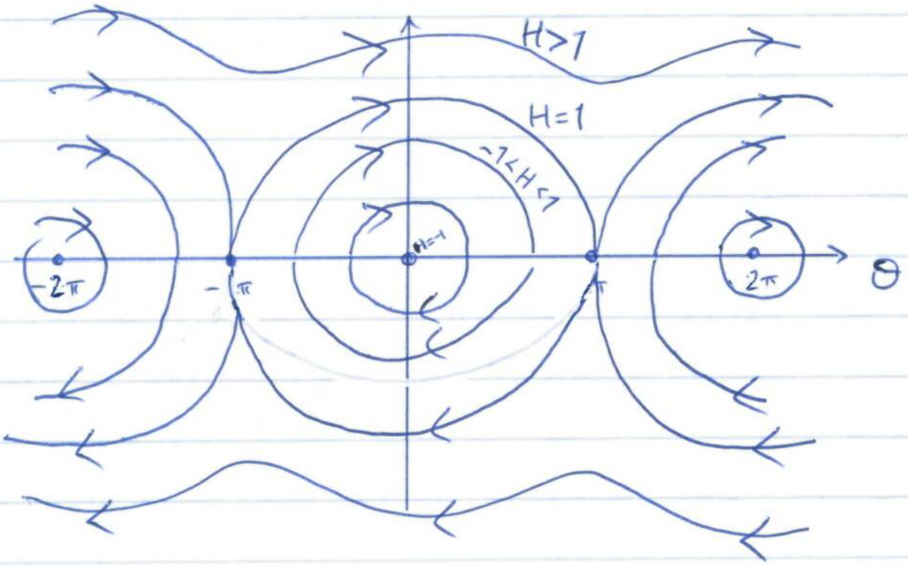
This is a Hamiltonian system (since $\dot{\theta}$ depends on v and vice-versa)

$$\dot{\theta} = \frac{\partial H}{\partial v} \qquad \dot{v} = -\frac{\partial H}{\partial \theta}$$

and get $H = \frac{1}{2}v^2 - \cos\theta$ (Hamiltonian is constant along the trajectory)

So the trajectories are ~~consts~~

$$v^2 = 2(H + \cos\theta) \qquad H \text{ const.}$$



$H < -1$, no trajectories

$H = -1$, trajectories consist of points $\theta = 2n\pi$, $v = 0$

$-1 < H < 1$, trajectories are closed and θ is restricted to lie in $(-\cos^{-1}(-H), +\cos^{-1}(H))$

$H > 1$, trajectories have either $v > 0$ or $v < 0$ and correspond to the pendulum performing

$H = 1$, $v^2 = 2(1 + \cos\theta)$

$$= 4\cos^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow v = \pm 2\cos\frac{\theta}{2} = \dot{\theta}$$

This can be integrated to show that θ takes infintely long to approach $\pm\pi (+2n\pi)$ as $t\to\infty$, and infintely long to approach $\mp\pi\,(+2n\pi)$ as $t\to-\infty$

Suppose (for $H=1$) the initial conditions are

$\theta=\theta_0$ (wlog $\theta_0\in(-\pi,\pi)$)

$\dot\theta = 2\left|\cos\frac{\theta_0}{2}\right|$ (wlog)

then $\gamma(\theta_0) = (-\pi,\pi)$ and $\alpha(\theta_0) = \{-\pi\}$
and $\omega(\theta_0) = \{\pi\}$.

Considered in the θ-v plane, this is a <u>heteroclinic orbit.</u>

Considered on the cylinder, this is a <u>homoclinic orbit.</u>

<u>Poincaré – Bendixon Theory in $\mathbb{R}^2$</u>

This is an elegant result in $\mathbb{R}^2$ which tells us that if a limit set doesn't contain any equilibria, then it must be a periodic orbit. This means that behaviour in $\mathbb{R}^2$ is much simpler than general behaviour in higher dimensions

[NOTE: The full theorem also deals with the possibilities where limit sets do contain equilibria and deals with 2D spaces other than $\mathbb{R}^2$. In fact, all possible limit sets in 2D are described, but we won't go into so much detail here.]

Theorem: Poincaré-Bendixson thm

Consider a C^1 vector field $\dot{x} = f(x)$ on $\mathbb{R}^2$. Suppose that the forward trajectory of a point x_0 enters a closed bounded region $E \subset \mathbb{R}^2$ and never leaves it (dun dun DUN!) Then $\omega(x_0)$ either contains an eq^{m} or is a periodic orbit.

This thm allows us to find periodic orbits for dynamical systems on $\mathbb{R}^2$ as follows:

- Find a 'trapping region' i.e. a region trajectories enter but never leave.
- If this region doesn't contain equilibria, or the eq^{a} it contains don't attract all trajectories, then it MUST contain a periodic orbit.

Our work last week tells us how to check if a region is a trapping region, e.g. if trajectories cross the boundary of the region inwards, then it is a trapping region.

Proof: not given.

Example of use The vector field
$$\dot{x} = \mu x - y - x(x^2+y^2)$$
$$\dot{y} = x + \mu y - y(x^2+y^2)$$

has a closed orbit for $\mu > 0$ because in polars:
$$\dot{r} = r(\mu - r^2)$$
$$\dot{\theta} = 1$$

The only eq^{m} is $r = 0$.

Let $\mu > 0$. Then on the circle, $r = \sqrt{\mu}/2$, we have $\dot{r} = 3\mu\sqrt{\mu}/8 > 0$.

On the circle $r = 2\sqrt{\mu}$, $\dot{r} = -6\mu\sqrt{\mu} < 0$. Thus these 2 circles bound a positively invariant region containing no equilibria.

Thus the region must contain a periodic orbit. In fact any two circles $r = r_1 < \sqrt{\mu}$ and $r = r_2 > \sqrt{\mu}$ bound an invariant region. Of course in this case it is also simple to find the periodic orbit directly from the DE in polars.

Exercise: Carry out the coord transform and write down the eq^n of the periodic orbit.

BASIC BIFURCATION THEORY

Broadly speaking, bifurcation theory is the theory of when we get <u>qualitative</u> changes in <u>families</u> of dynamical systems.

$$x_{n+1} = f(x_n, \mu)$$
$$\text{or} \quad \dot{x} = f(x, \mu).$$

For example, in the discrete case, the logistic family.

μ are parameters, and we want to know whether there are values of μ where something suddenly changes. This usually means 'birth, death or change of stability' of limit sets. To keep it simple, here we will say $\mu \in \mathbb{R}$.

Bifurcations fall into two categories:

<u>Local bifurcations</u> are changes which occur in some small neighbourhood of phase space. For example, an object might change stability or disappear. But outside the neighbourhood of the object, there may be no major changes.

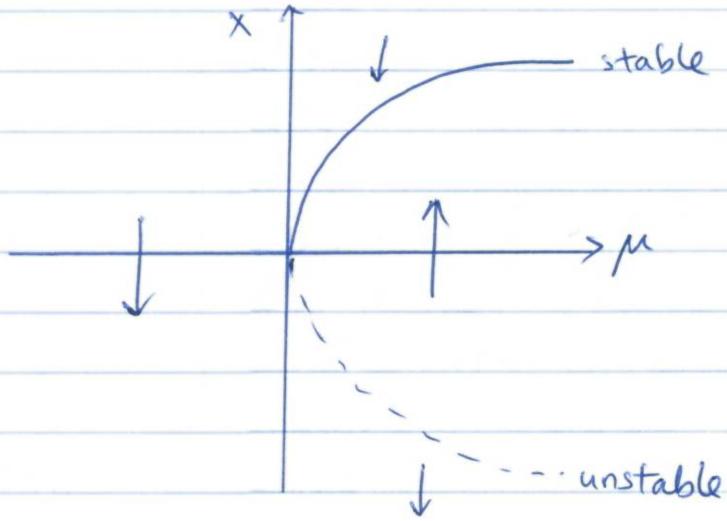
<u>global bifurcations</u> involve qualitative changes in the orbit structure in the dynamical system which are not restricted to a small area of phase space. Although they are fascinating, they are much harder to study. It is possible for limit sets which are not restricted to a small area of phase space (eg chaotic sets) to be born in such bifurcations

The Saddle-Node bifurcation

This is the only bifurcation we will look at in detail.
Consider the 1D ODE $\dot{x} = \mu - x^2$.

It is useful to plot the eq^m set defined by $0 = \mu - x^2$ in the μ-x plane.

At any fixed value of μ, this plot tells us about the equilibria at that value of μ.



We see that the point $(0,0)$ is a special point on this curve. For $\mu < 0$, there are no equilibria
for $\mu > 0$, there are 2.
↳ one at $x = \sqrt{\mu}$
↳ one at $x = -\sqrt{\mu}$

for $\mu = 0$, there is 1 nonhyperbolic eq^m.

Clearly at $\mu = 0$ an important 'event' occurs.
This event is a <u>saddle-node bifurcation</u>

<u>Note</u> that finding the eq^m for this ODE is the same as finding equilibria for the map

$$x_{n+1} = x_n + \mu - x_n^2.$$

If $\dot{x} = f(x,\mu) = \mu - x^2$, $\quad f'(x) = -2x$

$\Rightarrow$ the eq^{m} at $x = \sqrt{\mu}$ is stable when it exists, and
" " " $x = -\sqrt{\mu}$ is unstable " " " .

The plot of the eq^{m} value x vs the parameter μ is called a <u>bifurcation diagram</u>.

It is conventional to draw stable eq^{a} with solid lines ———
and unstable eq^{a} with dashed lines - - - .

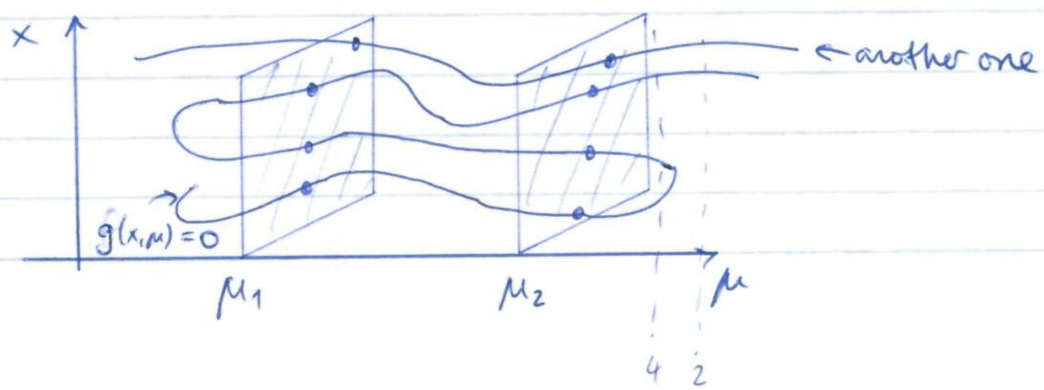
<u>The general argument</u>

If we have a one-parameter family of maps or flows, to find the fixed points we have to solve an equation of the form $g(x,\mu) = 0$, where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ and
$g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.

In the continuous case, g is just the RHS of the ODE system

For the discrete-time system, $x_{n+1} = f(x_n, \mu)$, we get $g(x,\mu) = f(x,\mu) - x$.

The zero set of a smooth f^{n} $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defines a set of <u>one-dimensional</u> curves in $\mathbb{R}^n \times \mathbb{R}$. If we fix $\mu = \mu_0$, we get a cross-section $\mathbb{R}^n \times \mu_0$. The 1D curves will, in general, intersect this cross-section in a set of points. These points will be sol^{n}s of $g(x,\mu_0) = 0$.



Characterisation of the SN bifurcation

Saddle-node bifurcations are points where a curve of solutions takes a 'U-turn' in the μ-dir^n.

In 1D, it's easy to write down sufficient conditions on $g(x,\mu)$ at points where it takes a U-turn in the μ-dir^n.
Note that the slope of the curve becomes infinite; or rotating the picture and thinking of μ as a f^n of x, the slope becomes 0. This means we can describe the curve near a SN bifurcation by thinking of μ as a f^n of x.
Then the slope of the curve $\mu = \mu(x)$ is obtained by differentiating $g(x, \mu(x)) = 0$, wrt x along the curve, with notation $g_x = \partial g/\partial x$, and $g_\mu = \partial g/\partial \mu$ etc. we get

$$\underbrace{g_x + g_\mu \frac{d\mu}{dx}}_{dg/dx} = 0$$

Which gives $\frac{d\mu}{dx} = -\frac{g_x}{g_\mu}$

So $\frac{d\mu}{dx} = 0$ if $g_x = 0$, as long as $g_\mu \neq 0$.

The first condition, $\underline{g_x = 0}$, is our main bifurcation condition

The second condition, $\underline{g_\mu \neq 0}$ is called a <u>genericity</u> condition.

If $g_\mu = 0$ then we cannot guarantee that $\frac{d\mu}{dx} = 0$, in fact the question may not make any sense because there may be no curve of sol^{n}s at all locally.

To get a true turning point $\frac{d^2\mu}{dx^2} \neq 0$ (sufficient but not necessary).

We can calculate the second derivative by differentiating $g(x, \mu(x)) = 0$ twice wrt x

$$g_{xx} + g_{x\mu}\frac{d\mu}{dx} + \frac{d^2\mu}{dx^2} g_\mu + \frac{d\mu}{dx} g_{x\mu} + \left(\frac{d\mu}{dx}\right)^2 g_{\mu\mu} = 0$$

At the bifurcation, $\frac{d\mu}{dx} = 0$, so $g_{xx} + g_\mu \frac{d^2\mu}{dx^2} = 0$

This gives $\frac{d^2\mu}{dx^2} = -\frac{g_{xx}}{g_\mu}$.

Since we have already assumed that $g_\mu \neq 0$, we see that $\frac{d^2\mu}{dx^2} = 0$ iff $g_{xx} = 0$.

Using only geometrical ideas, we have found four conditions which together mean that the eq^n $g(x, \mu) = 0$ has a SN bifurcation at (x_0, μ_0). These are:

1. $g(x_0, \mu_0) = 0$ ($\exists$ an eq^m at (x_0, μ_0))

2. $g_x(x_0, \mu_0) = 0$ (condition for nonhyperbolicity)

3. $g_\mu(x_0, \mu_0) \neq 0$ (a genericity condition)

4. $g_{xx}(x_0, \mu_0) \neq 0$ (another genericity condition)

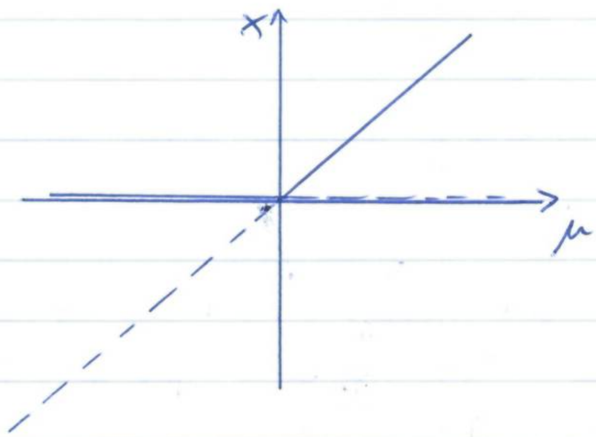
Completing the argument: the Implicit Function Thm (IFT)

Above we have not proved that there must be a curve of sol^{n}s $\mu = \mu(x)$ (near (x_0, μ_0)) or that it must be unique. The IFT guarantees that since $g_\mu \neq 0$, there is a unique curve of sol^{n}s defined near the point of interest.

<u>Example:</u> $\dot{x} = x(\mu - x) = g(x,\mu)$, $g(0,0) = 0$.

<u>Not</u> SN bif$^{\underline{n}}$ because it fails (3).

This is actually a nongeneric bifurcation called a transcritical bifurcation — in which 2 sol$^{\underline{n}}$s meet, exchange stability and diverge.



<u>The SN bif$^{\underline{n}}$ in higher dimensions</u>

The basic conditions are easy to state:

(1) There must be a sol$^{\underline{n}}$ at (x_0, μ_0)
i.e. $g(x_0, \mu_0) = 0$.

(2) The Jacobian at (x_0, μ_0) must have a zero eigenvalue
i.e. $\det(Dg(x_0, \mu_0)) = 0$

The genericity conditions have the same geometrical meaning as in 1D but are complicated to state/check.

Bifurcations

There are two scenarios for <u>flows</u> (ODEs) :

(1) A single real eigenvalue of the Jacobian passes through zero — a SN bif^n if gen. conditions are satisfied

(2) A pair of complex conj. eigenvalues pass through the imaginary axis — a Hopf bifurcation

There are three scenarios for <u>maps</u> :

(1) A single real eigenvalue of the Jacobian passes through 1 — a SN bif^n

(2) A single real eigenvalue of the Jacobian passes through -1 — a period-doubling bif^n

(3) A pair of complex conj. eigenvalues pass through the unit circle — a Hopf bifurcation.

Example :
$$\dot{x} = \mu - x^2 \qquad \dot{y} = -y \qquad \left.\begin{matrix}(\sqrt{\mu}, 0) \\ (-\sqrt{\mu}, 0)\end{matrix}\right\} \text{fixed pts.}$$

look for bif^n at origin.

Examples of Poincaré - Bendixson

2. $\dot{x} = x - y - x(x^2+y^2) + \frac{x^2 y}{2}$

$\dot{y} = x + y - y(x^2+y^2) + \frac{y^2 x}{2}$

What do trajectories do as $t \to \infty$?

$$r^2\dot{\theta} = y\dot{x} - x\dot{y} = \cancel{yx} - y^2 - \cancel{xy(x^2+y^2)} + \cancel{\frac{x^2y^2}{2}}$$
$$-x^2 - \cancel{xy} + \cancel{xy(x^2+y^2)} - \cancel{\frac{x^2y^2}{2}}$$

$$= -(x^2+y^2)$$

$\dot{\theta} = -1 \Rightarrow$ only equilibrium is at $x=0,\ y=0$

$V = x^2+y^2 \quad (V \geq 0)$

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x^2 - 2xy - 2x^2(x^2+y^2) + x^3y$$
$$+ 2xy + 2y^2 - 2y^2(x^2+y^2) + y^3x$$

$$= 2V - 2V^2 + xyV$$

If $V = 4 \Rightarrow \dot{V} = 8 - 32 + 4xy$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \leq 8$

$(2(x-y)^2 \geq 0 \quad 2(x^2+y^2) \geq 4xy$

So $\dot{V} < 0$

$$V = x^2 + y^2 = 1/4$$

$$\Rightarrow \dot{V} = \frac{1}{2} - \frac{1}{8} + \frac{xy}{4} \geqslant -1/8$$

$$\left(2(x+y)^2 \geqslant 0 \Rightarrow 4xy \geqslant -2(x^2+y^2) \right)$$

The region $1/4 < x^2 + y^2 < 4$ is forward invariant & contains no equilibrium $\xRightarrow{\text{PB theorem}}$ $\exists$ a limit cycle in $1/4 < x^2 + y^2 < 4$

could have used $V > \frac{4}{3}$, $\dot{V} < 0$

$V > 4/5$, $\dot{V} > 0$

All i.c.s except $x_0 = y_0 = 0$ have $\omega(x_0, y_0)$ a periodic orbit

Example

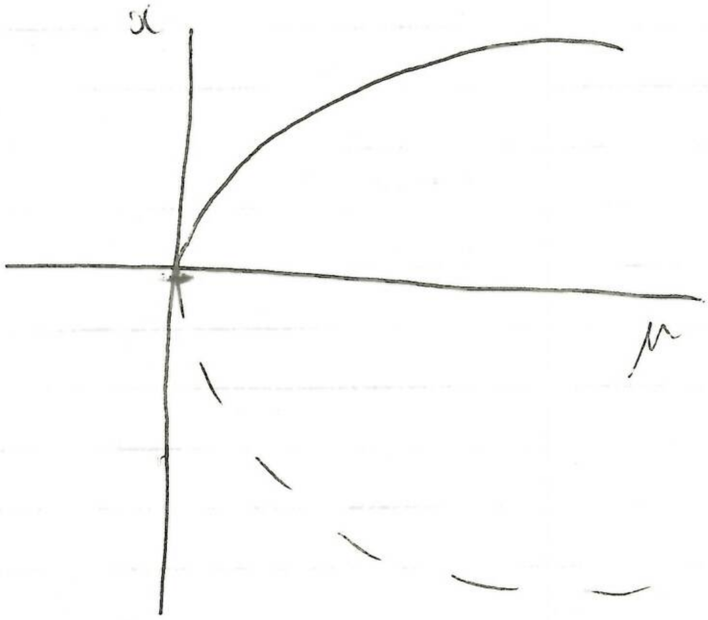
$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$

Steady states are $(\sqrt{\mu}, 0)$ & $(-\sqrt{\mu}, 0)$ for $\mu \geqslant 0$

$$J = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix} \qquad J\Big|_{(\sqrt{\mu}, 0)} = \begin{pmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix}$$

E-values -1, $-2\sqrt{\mu}$
Stable node

$$J\Big|_{(-\sqrt{\mu}, 0)} = \begin{pmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \text{saddle unstable}$$



$\mu = 0$ is a SN bifurcation

When $\mu = 0$ $(0,0)$ the Jacobians have a zero eigenvalue

The point $(0,0,0)$ is a fixed point & the Jacobian has a zero eigenvalue $\Rightarrow$ the main conditions for a bifurcation occur.